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Determinants of matrices over commutative finite principal ideal rings ☆

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ABSTRACT

In this paper, the determinants of $n \times n$ matrices over commutative finite chain rings and over commutative finite principal ideal rings are studied. The number of $n \times n$ matrices over a commutative finite chain ring R of a fixed determinant a is determined for all $a \in R$ and positive integers n. Using the fact that every commutative finite principal ideal ring is a product of commutative finite chain rings, the number of $n \times n$ matrices of a fixed determinant over a commutative finite principal ideal ring is shown to be multiplicative, and hence, it can be determined. These results generalize the case of matrices over the ring of integers modulo m.

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1. Introduction

Determinants are known for their applications in matrix theory and linear algebra, e.g., determining the area of a triangle via Heron's formula in [8], solving linear systems using Cramer's rule in [3], and determining the singularity of a matrix. Therefore, properties of matrices and determinants of matrices have been extensively studied (see [3], [12], and references therein). Especially, matrices over finite fields are interesting due to their rich algebraic structures and various applications. Singularity of such matrices is useful in applications. For example, nonsingular matrices over finite fields are good choices for constructing good linear codes in [1]. The number of $n \times n$ singular (resp., nonsingular) matrices over a finite field $\mathbb{F}_q$ was studied in [13]. As a generalization of the prime field $\mathbb{Z}_p$, the determinants of matrices over the ring $\mathbb{Z}_m$ of integers modulo m were studied in [2] and [11]. The number of $n \times n$ matrices over $\mathbb{Z}_m$ of a fixed determinant has been first studied in [2]. In [11], a different and simpler technique was applied to determine the number of such matrices over $\mathbb{Z}_m$.

Commutative finite principal ideal rings (CFPIRs), a generalization of the ring of integers modulo m, are interesting since they have applications in many branches of mathematics and links to other objects. Cyclic codes of length n over the finite field $\mathbb{F}_q$ are identified with the ideals in the principal ideal ring $\mathbb{F}_q[x]/\langle x^n - 1\rangle$ (see [10]). The ring of $n \times n$ circulant matrices over fields is a principal ideal ring (see [9]). Some nonsingular matrices over a CFPIR have been applied in constructing good matrix product codes in [4]. Therefore, the determinants of matrices over CFPIRs are interesting.

To the best of our knowledge, the enumeration of $n \times n$ matrices of a fixed determinant over CFPIRs has not been completed. It is therefore of natural interest to determine the number $d_n(\mathcal{R}, r)$ of $n \times n$ matrices of determinant r over a CFPIR $\mathcal{R}$. Note that every CFPIR $\mathcal{R}$ is a product of commutative finite chain rings (CFCRs). This property allows us to separate the study into two steps: 1) determine the number $d_n(R, a)$ of $n \times n$ matrices over a CFCR R whose determinant is a for all $n \in \mathbb{N}$ and $a \in R$, and 2) show that the number $d_n(\mathcal{R}, r)$ is multiplicative among the isomorphic components of r. The number $d_n(\mathcal{R}, r)$ is therefore follows.

The paper is organized as follows. In Section 2, some definitions and properties of rings and matrices are recalled. In Section 3, the number $d_n(R, a)$ of $n \times n$ matrices over a CFCR R having determinant a is determined for all $a \in R$ and $n \in \mathbb{N}$. In Section 4, using the fact that every CFPIR is isomorphic to a product of CFCRs and results in Section 3, the number $d_n(\mathcal{R}, r)$ of $n \times n$ matrices over a CFPIR $\mathcal{R}$ having determinant r is determined for all $r \in \mathcal{R}$ and $n \in \mathbb{N}$.

2. Preliminaries

In this section, definitions and some properties of rings and matrices are recalled.

A ring $\mathcal{R}$ with identity $1 \neq 0$ is called a *commutative finite principal ideal ring (CFPIR)* if $\mathcal{R}$ is finite commutative and every ideal of R is principal. A ring R is called