Finite Fields and Their Applications 48 (2017) 227–240

Contents lists available at ScienceDirect

Finite Fields and Their Applications

www.elsevier.com/locate/ffa

Algebraic Cayley graphs over finite local rings

Arnisa Rasri, Yotsanan Meemark *

Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University, Bangkok, 10330, Thailand

ARTICLE INFO

Article history:
Received 3 February 2017
Received in revised form 28 May 2017
Accepted 4 August 2017
Available online xxxx
Communicated by Daqing Wan

MSC:
11L40
05C75

Keywords:
Algebraic Cayley graphs
Expander graphs
Weil's bound

ABSTRACT

In this work, we define and study the algebraic Cayley directed graph over a finite local ring. Its vertex set is the unit group of a finite extension of a finite local ring R and its adjacency condition is that the quotient is a monic primary polynomial. We investigate its connectedness and diameter bound, and we also show that our graph is an expander graph. In addition, if a local ring has nilpotency two, then we obtain a better view of our graph from the lifting of the graph over its residue field.

1. Introduction

Let R be a finite commutative local ring with maximal ideal M and residue field $\Bbbk = R/M$ equipped with the reduction map $^{-} : R \longrightarrow \Bbbk$. A polynomial in $R[x]$ is *regular* if its reduction is not zero in the residue field. Let $f(x)$ be a primary regular non-unit polynomial in $R[x]$. That is, its reduction $\overline{f}(x)$ is a power of an irreducible polynomial in $\Bbbk[x]$. By Hensel's lemma,

* Corresponding author.
E-mail addresses: arnisarasri@gmail.com (A. Rasri), yotsanan.m@chula.ac.th (Y. Meemark).

http://dx.doi.org/10.1016/j.ffa.2017.08.002
1071-5797/

$$f(x) = \delta(x)\pi(x)^s + \beta(x)$$

where $s \in \mathbb{N}, \delta(x)$ is a unit, $\beta(x) \in M[x]$ and $\pi(x)$ is a monic irreducible polynomial in $R[x]$ of degree $n \in \mathbb{N}$ such that its reduction $\overline{\pi}(x)$ is irreducible in $\Bbbk[x]$. Let I denote the principal ideal generated by $f(x)$. It follows from Theorem 4.3.2 of [1] that $R[x]/I$ is a local ring of order $|R|^{ns}$ with maximal ideal

$$\begin{aligned}\mathcal{M} &= \langle M, \pi(x)\rangle/I \\ &= \{h(x) + \pi(x)g(x) + I : h(x) \in M[x], g(x) \in R[x], \deg h(x) < n, \\ &\qquad\qquad \deg g(x) < n(s-1)\},\end{aligned}$$

so that $|\mathcal{M}| = |M|^n |R|^{n(s-1)}$. Moreover, the quotient field

$$R[x]/I \Big/ \langle M, \pi(x)\rangle/I \cong R[x]/\langle M, \pi(x)\rangle \cong \Bbbk[x]/\langle \overline{\pi}(x)\rangle,$$

and the canonical map is given by

$$\mu(g(x) + I) = \overline{g}(x) + \langle \overline{\pi}(x)\rangle$$

for all $g(x) \in R[x]$. Write $\Gamma_f(R)$ for the unit group $(R[x]/I)^\times$. Then

$$|\Gamma_f(R)| = |R|^{ns} - |\mathcal{M}| = |R|^{n(s-1)}(|R|^n - |M|^n).$$

Assume that $n > 1$. For $1 \le d < n$, let $P_d(R)$ be the set of monic primary polynomials of degree d in $R[x]$. Define the Cayley graph $G_d(R, f)$ as follows. The vertex set is $\Gamma_f(R)$ and there is a directed edge from $g_1(x) + I$ to $g_2(x) + I$ if and only if $(g_2(x) + I)(g_1(x) + I)^{-1}$ is in $P_d(R) + I$, the set of cosets of I with representatives in $P_d(R)$. This graph depends on three parameters, namely, d, R, f. It is a regular graph of degree $|P_d(R)|$. Since $d < n$, $P_d(R) + I$ is a subset of $\Gamma_f(R)$ and $|P_d(R) + I| = |P_d(R)|$. Note that the cardinality of $P_d(R)$ is equal to the number of the liftings of monic primary polynomials of degree d in $\Bbbk[x]$. Hence,

$$|P_d(R)| = |M|^d |P_d(\Bbbk)|,$$

where $P_d(\Bbbk)$ is the set of monic primary polynomials of degree d in $\Bbbk[x]$. Moreover,

$$|P_d(\Bbbk)| = \sum_{l|d} \frac{1}{l} \sum_{m|l} \mu(m)|\Bbbk|^{\frac{l}{m}} \sim \frac{|\Bbbk|^d}{d},$$

where μ is the Möbius function. If R is a finite field and $f(x)$ is an irreducible polynomial, then the graph $G_d(R, f)$ reduces to $G_d(\Bbbk, \overline{\pi})$, which has been studied by Lu et al. [4]. Their idea came from Chung's graphs $G_1(\Bbbk, \overline{\pi})$ [2]. Recently, Huang and Liu [3]