Finite Fields and Their Applications 48 (2017) 241–260

Contents lists available at ScienceDirect

Finite Fields and Their Applications

www.elsevier.com/locate/ffa

On the structure of $\mathbb{Z}_2\mathbb{Z}_2[u^3]$-linear and cyclic codes ☆

Ismail Aydogdu [a,*], Irfan Siap [b], Roger Ten-Valls [c]

[a] *Department of Mathematics, Yildiz Technical University, Istanbul, Turkey*
[b] *Jacodes Math Institute, Istanbul, Turkey*
[c] *Department of Information and Communication Engineering, Universitat Autònoma de Barcelona, Bellaterra, Spain*

A R T I C L E   I N F O

Article history:
Received 8 December 2015
Received in revised form 11 February 2017
Accepted 3 March 2017
Available online xxxx
Communicated by W. Cary Huffman

MSC:
94B05
94B60

Keywords:
Linear codes
Cyclic codes
$\mathbb{Z}_2\mathbb{Z}_2[u^3]$-linear cyclic codes
Duality

A B S T R A C T

Recently some special type of mixed alphabet codes that generalize the standard codes has attracted much attention. Besides $\mathbb{Z}_2\mathbb{Z}_4$-additive codes, $\mathbb{Z}_2\mathbb{Z}_2[u]$-linear codes are introduced as a new member of such families. In this paper, we are interested in a new family of such mixed alphabet codes, i.e., codes over $\mathbb{Z}_2\mathbb{Z}_2[u^3]$ where $\mathbb{Z}_2[u^3] = \{0, 1, u, 1 + u, u^2, 1 + u^2, u + u^2, 1 + u + u^2\}$ is an 8-element ring with $u^3 = 0$. We study and determine the algebraic structures of linear and cyclic codes defined over this family. First, we introduce $\mathbb{Z}_2\mathbb{Z}_2[u^3]$-linear codes and give standard forms of generator and parity-check matrices and later we present generators of both cyclic codes and their duals over $\mathbb{Z}_2\mathbb{Z}_2[u^3]$. Further, we present some examples of optimal binary codes which are obtained through Gray images of $\mathbb{Z}_2\mathbb{Z}_2[u^3]$-cyclic codes.

☆ This work has been partially supported by the Spanish MINECO grants TIN2016-77918-P(AEI/FEDER, UE) and MTM2015-69138-REDT, and by the Catalan grant 2014SGR691.

* Corresponding author.
E-mail addresses: iaydogdu@yildiz.edu.tr (I. Aydogdu), irfan.siap@gmail.com (I. Siap), rten@deic.uab.cat (R. Ten-Valls).

http://dx.doi.org/10.1016/j.ffa.2017.03.001
1071-5797/© 2017 Published by Elsevier Inc.

1. Introduction

Linear codes over finite rings have attracted a great attention after the famous paper written by Hammons et al. in 1994 [11] where algebraic structures are presented for some well-known nonlinear binary codes via a Gray map. After this paper, there have been and are still many studies on codes over rings. Recently, codes over mixed alphabet rings viewed as submodules have been studied. The first of these studies is an interesting paper authored by Borges et al. (2010) presenting $\mathbb{Z}_2\mathbb{Z}_4$-additive codes as $\mathbb{Z}_4$ submodules (additive groups) of $\mathbb{Z}_2^{\alpha} \times \mathbb{Z}_4^{\beta}$ where α and β are positive integers [7]. Later, Aydogdu and Siap generalized these additive codes to codes over $\mathbb{Z}_2 \times \mathbb{Z}_{2^s}$ [3] and $\mathbb{Z}_{p^r} \times \mathbb{Z}_{p^s}$ [4] where r and s $(1 \leq r < s)$ are positive integers and p is prime. The mixed alphabet approach has brought other possible choices and also new directions to be explored. In one of the such studies, Aydogdu et al. have introduced $\mathbb{Z}_2\mathbb{Z}_2[u]$ codes where $\mathbb{Z}_2[u] = \{0, 1, u, 1+u\}$ and $u^2 = 0$ as submodules in [5] recently. Although, the structure of these codes is similar to the structure of codes over $\mathbb{Z}_2 \times \mathbb{Z}_4$, these codes have some advantages compared to $\mathbb{Z}_2\mathbb{Z}_4$-additive codes. For example, the Gray images of linear codes over $\mathbb{Z}_2\mathbb{Z}_2[u]$ are also binary linear codes, however, this is not always the case for codes over $\mathbb{Z}_2\mathbb{Z}_4$. Another advantage of working over such submodules is that, the factorization of polynomials in $\mathbb{Z}_2[x]$ is also valid since $\mathbb{Z}_2$ is a subring of $\mathbb{Z}_2[u]$ and Hensel's lift is not necessary.

Recently, Abualrub et al. defined cyclic codes for $\mathbb{Z}_2\mathbb{Z}_4$-additive codes [2]. Inspired by this paper, Aydogdu et al. presented the structure of cyclic and constacyclic codes and their duals in [6].

In this paper we generalize the results of the papers [5] and [6] as $\mathbb{Z}_2\mathbb{Z}_2[u^3]$-linear and cyclic codes and determine the spanning sets of both cyclic codes and their duals. We also give some examples of optimal binary codes derived from the $\mathbb{Z}_2\mathbb{Z}_2[u^3]$-cyclic codes.

2. $\mathbb{Z}_2\mathbb{Z}_2[u^3]$-linear codes

Consider the finite binary field $\mathbb{Z}_2 = \{0, 1\}$ and the finite ring $\mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2 = \mathcal{R}_3 = \left\{0, 1, u, 1+u, u^2, 1+u^2, u+u^2, 1+u+u^2\right\}$ where $u^3 = 0$. It is clear that the ring $\mathbb{Z}_2$ is a subring of $\mathcal{R}_3$. We construct the set

$$\mathbb{Z}_2\mathcal{R}_3 = \{(v, v') \,|v \in \mathbb{Z}_2 \text{ and } v' \in \mathcal{R}_3\}.$$

This set can not be made an $\mathcal{R}_3$-submodule with respect to scalar multiplication directly. We need to define an auxiliary map

$$\eta : \mathcal{R}_3 \to \mathbb{Z}_2 \tag{1}$$
$$\eta\left(a + ub + u^2c\right) = a,$$