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A strongly regular decomposition of the complete graph and its association scheme

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ABSTRACT

For any positive integer m, the complete graph on $2^{2m}(2^m+2)$ vertices is decomposed into 2^m+1 commuting strongly regular graphs, which give rise to a symmetric association scheme of class $2^{m+2}-2$. Furthermore, the eigenmatrices of the symmetric association schemes are determined explicitly. As an application, the eigenmatrix of the commutative strongly regular decomposition obtained from the strongly regular graphs is derived.

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1. Introduction

A *strongly regular graph with parameters* (v, k, λ, μ) is a regular graph with v vertices of degree k such that every two adjacent vertices have exactly λ common neighbors and every two non-adjacent vertices have exactly μ common neighbors. A *strongly regular decomposition* is a decomposition of the edge set of the complete graph with vertex set V into strongly regular graphs with vertex set V. A strongly regular decomposition is *commutative* if the adjacency matrices of strongly regular graphs are commutative. The concept of strongly regular decomposition was introduced by van Dam [2] in order to study more general situation of amorphous association schemes.

In this paper, we show that for any positive integer m, there is a commutative strongly regular decomposition of the complete graph of $2^{2m}(2^m + 2)$ vertices, into 2^m strongly regular graphs with parameters $(2^{2m}(2^m + 2), 2^{2m} + 2^m, 2^m, 2^m)$ and $2^m + 2$ cliques of size 2^{2m}. Note that $2^m + 2$ cliques of size 2^{2m} is a strongly regular graph with parameters $(2^{2m}(2^m + 2), 2^{2m} - 1, 2^{2m} - 2, 0)$. In fact, the constructed strongly regular graphs with parameters $(2^{2m}(2^m + 2), 2^{2m} + 2^m, 2^m, 2^m)$ are symmetric $(2^{2m}(2^m + 2), 2^{2m} + 2^m, 2^m)$-designs with symmetric incidence matrices with very large number of symmetries. Our construction method is due to Wallis [5] (see also [4, Theorem 5.23]) based on the generalized Hadamard matrices obtained from finite fields of characteristic two and symmetric Latin squares with constant diagonal of even order.

One might wonder if the decomposition yields a symmetric association scheme, but unfortunately this does not hold. We had to further decompose the edge sets of the strongly regular graphs in order to obtain a symmetric association scheme, and to determine the eigenmatrices of the symmetric association scheme explicitly. As a corollary, we obtain the eigenmatrix of the commutative strongly regular decomposition.

2. Preliminaries

Let n be a positive integer. Let V be a finite set of size v and R_i ($i \in \{0, 1, \dots, n\}$) be a non-empty subset of $V \times V$. The *adjacency matrix* A_i of the graph with vertex set V and edge set R_i is a $v \times v$ $(0, 1)$-matrix with rows and columns indexed by the elements of V such that $(A_i)_{xy} = 1$ if $(x, y) \in R_i$ and $(A_i)_{xy} = 0$ otherwise. The pair $(V, \{R_i\}_{i=0}^n)$ is said to be a *commutative decomposition* of the complete graph if the following hold:

(i) $A_0 = I_v$, the identity matrix of order v.
(ii) $\sum_{i=0}^n A_i = J_v$, the all-ones matrix of order v.
(iii) A_i is symmetric for $i \in \{1, \dots, n\}$.
(iv) For any i, j, $A_i A_j = A_j A_i$.

We also refer to the set of non-zero $v \times v$ $(0, 1)$-matrices satisfying (i)-(iv) as a commutative decomposition. Note that the corresponding graph of each A_i is regular, because A_i and $J_v = \sum_{i=0}^n A_i$ commute. Let k_i denote the valency of the corresponding graph of A_i.