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G_2 and some exceptional Witt vector identities

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ABSTRACT

We find some new one-parameter families of exponential sums in every odd characteristic whose geometric and arithmetic monodromy groups are G_2.

0. Introduction

In earlier work [7, 9.1.1], we proved that certain very simple one-parameter families of exponential sums had the exceptional group G_2 as their (geometric and arithmetic) monodromy groups, in every finite characteristic $p \geq 17$. These sums were of the form

$$(1/g) \sum_{x \in k^\times} \chi_2(x)\psi(x^7 + tx).$$

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Here k is a finite field, g is a fixed gauss sum, χ_2 is the quadratic character of $k^\times$, ψ is a nontrivial additive character of k, and $t \in k$ is the parameter. A question of Rudnick and Waxman led us to wonder if, in this construction, the polynomial x^7 inside the ψ could be replaced by other polynomials of degree seven and still yield G_2. Computer experiments suggested that the answer was indeed yes, for polynomials of the form

$$ax^7/7 + 2abx^5/5 + ab^2x^3/3,$$

any $a \neq 0$, any b. That these polynomials do indeed produce G_2 in large characteristic (see Theorem 4.3) results from certain Witt vector identities. It remains an open question if these are the only polynomials which produce G_2.

In the second half of the paper, we analyze the situation in low characteristic, especially in characteristics $3, 5, 7$, where Witt vectors reappear in order to make sense of the question, and (again) to provide the answer.

1. The exceptional identities

Fix a prime p, and consider the p-Witt vectors of length 2 as a ring scheme over $\mathbb{Z}$. The addition law is given by

$$(x, a) + (y, b) := (x + y, a + b + (x^p + y^p - (x + y)^p)/p).$$

The multiplication law is given by

$$(x, a)(y, b) := (xy, x^p b + y^p a + pab).$$

For an odd prime p, we have

$$(x, 0) + (y, 0) + (-x - y, 0) = (0, (x^p + y^p - (x + y)^p)/p).$$

Let us define, for odd p, the integer polynomial

$$F_p(x, y) := (x^p + y^p - (x + y)^p)/p \in \mathbb{Z}[x, y].$$

For $p = 2$, we have

$$(x, 0) + (y, 0) + (-x - y, 0) = (0, x^2 + xy + y^2),$$

and we define

$$F_2(x, y) := x^2 + xy + y^2 \in \mathbb{Z}[x, y].$$

Thus

$$F_3 = -xy(x + y).$$