Finite Fields and Their Applications 44 (2017) 18–21

Contents lists available at ScienceDirect

Finite Fields and Their Applications

www.elsevier.com/locate/ffa

Multiple Hamilton cycles in bipartite cubic graphs: An algebraic method

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A R T I C L E I N F O

Article history:
Received 22 June 2015
Received in revised form 21 January 2016
Accepted 5 November 2016
Available online xxxx
Communicated by L. Storme

MSC:
05C45
94B05
14G15
51E99

Keywords:
Graph
Cubic
Hamilton cycle
Bipartite
Determinant
Finite field
GF(2)

A B S T R A C T

Many important graphs are bipartite and cubic (i.e. bipartite and trivalent, or "bicubic"). We explain concisely how the Hamilton cycles of this type of graph are characterized by a single determinantal condition over GF(2). Thus algebra may be used to derive results such as those of Bosák, Kotzig, and Tutte that were originally proved differently.

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http://dx.doi.org/10.1016/j.ffa.2016.11.006
1071-5797/

1. Introduction

Most commonly, combinatorial methods are used to prove results in graph theory; see e.g. [5]. However, it is also possible to use geometrical/matroid theoretic methods as in pioneering work by Tutte; see [1]. Here we look at a different method using an algebraic equation derived from a determinant.

In this note the graphs G are all simple, undirected, cubic and bipartite, with vertex set V and edge set E. Thus $|V| = v = 2r$ with bipartition $V = A \cup B$, where $A = \{v_1, \ldots, v_r\}$, $B = \{w_1, \ldots, w_r\}$. The $e = 3r$ edges $[v_i, w_j]$ of E go between $v_i \in A$ and $w_j \in B$; see [3]. Small such graphs are $K_{3,3}$, the complete bicubic graph, often called the *utility graph*; and the cubical graph consisting of the 8 vertices and 12 edges on a cube.

A "cycle" in a cubic graph is a collection of edges with an even number of (i.e. 0 or 2 here) edges through each vertex. Each cycle is the union of connected parts, which are called "circuits" (when non-empty). This terminology is from matroid theory, where edges of a circuit are the same as a minimal dependent set of points in the "circuit-matroid" of the graph. It is clear that the sum (mod 2), or taking symmetric differences, of any set of cycles is also a cycle. So there is a binary "cycle-code" based on the edges of the graph. The vertices of a cycle are those on two edges of it. So, for cubic graphs, the decomposition of a cycle into circuits is "vertex-disjoint", since a pair of circuits with a common vertex would have a common edge, $2 + 2$ being greater than 3. For a general graph G this cycle code has parameters $[e, e - v + 1]_2$ (length e, and dimension $e - v + 1$). To see this, each vertex gives a binary linear condition, but the sum of the v linear conditions is zero. So in the case of cubic graphs which have v vertices and $e = 3v/2$ edges, the parameters are $[e, v/2 + 1]_2$. We consider only bipartite cubic graphs in this paper, and so $v = 2r$ and $e = 3r$, while the binary cycle code has parameters $[e, r + 1]_2$. In this case the circuits (and also the cycles) all have even size, and the cycle code is an "even" code (every word has even weight).

Any finite geometrical configuration (or linear space) of m points and n lines has a bipartite (Levi) graph with $m + n$ vertices, each edge of the graph corresponding to an incidence between a point and a line. Conversely, in the case we are considering, a bicubic graph G has girth at least 6, having no circuits of size 4 or smaller, if and only if it is the Levi graph of a linear space of r points and r lines, with 3 lines through each point, and 3 points on each line. This is by definition an r_3-configuration in geometry.

Examples include the Heawood graph on 7 vertices, corresponding to the projective plane of order 2 or the unique 7_3; the Möbius–Kantor graph on 16 vertices, corresponding to the unique 8_3; the Pappus graph on 18 vertices, corresponding to Pappus configuration, one of the three possible 9_3; the Desargues graph on 10 vertices, corresponding to Desargues configuration, the best known of the ten possible 10_3; the girth 8 Tutte–Coxeter graph on 30 vertices, corresponding to the generalized quadrangle W_2 of 15 points and lines fixed by a symplectic polarity in 3-d space $\mathrm{PG}(3, 2)$ over $\mathrm{GF}(2)$. Now we consider the general bicubic graph G.