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Explicit Galois representations of automorphisms on holomorphic differentials in characteristic p

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ABSTRACT

We determine the representation of the Galois group for the cyclotomic function fields in characteristic $p > 0$ induced by the natural action on the space of holomorphic differentials via construction of an explicit canonical basis of differentials. This includes those cases which present wild ramification and finite automorphism groups with non-cyclic p-part, which have remained elusive. We also obtain information on the gap sequences of ramified primes and extend these results to rank one Drinfel'd modules.

1. Introduction

Let K denote an algebraic function field with constant field k and genus g_K, where $g_K \geq 2$. The space Ω_K of holomorphic differentials of K is a vector space over k and

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represents the finite group of automorphisms G of K/k, as introduced by Hurwitz [13]. If char $k = 0$, the representation of G induced by Ω_K was completely determined by Chevalley and Weil [3]. In char $k > 0$, this problem remains open. Various methods have been introduced to solve this problem with certain assumptions on ramification or group structure, but in general the structure of Ω_K as a representation space for G is not known. In this section, we briefly review previous work on this problem and summarise our approach to an explicit solution in positive characteristic, which has its origins in class field theory.

The construction of Chevalley and Weil may be completed algebraically by evaluating the different of the extension K/K^G. Explicitly:

Theorem *(Chevalley–Weil, 1932). Let $f : X \to Y$ be a Galois covering of (projective, non-singular) curves defined over $\mathbb{C}$ with Galois group G. Let $|G| = n$ and ζ_n be a primitive nth root of unity. Let χ be an irreducible representation of G. Then the multiplicity of ν_χ of χ in $H^0(X, \Omega_X)$ is equal to*

$$v_\chi = d_\chi(g_Y - 1) + \sum_{\mu=1}^{r} \sum_{\alpha=0}^{e_\mu - 1} N_{\mu\alpha} \left\langle -\frac{\alpha}{e_\mu} \right\rangle + \sigma.$$

In this formula,

- *d_χ denotes the degree of the representation ρ_χ associated with χ,*
- *g_Y the genus of Y,*
- *e_μ the ramification index at a (ramified) prime μ,*
- *$N_{\mu\alpha}$ the multiplicity of the eigenvalue $\zeta_n^{n\alpha/e_\mu}$ of the representation of the inertia group at μ induced by ρ_χ, and*
- *$\sigma \in \{0, 1\}$ with $\sigma = 1$ if, and only if, χ is trivial.*

Via Grothendieck, this extends to all cases in char $k = p > 0$ where $(|G|, p) = 1$ using a lifting argument to characteristic 0, which then allows application of Chevalley–Weil (see also [27]). It is worthy of note that Tamagawa had also resolved the unramified cyclic case [24], using Hasse and Witt's generation of unramified extensions of degree p to show that

$$\Omega_K \cong (k[G])^{g_{K^G} - 1} \oplus I_G,$$

where the former component denotes $g_{K^G} - 1$ copies of $k[G]$ and K^G the fixed field of G. Tamagawa conjectured that this was likely true for all non-cyclic extensions in positive characteristic, which is not correct; using the Frattini subgroup, Hasse–Witt theory, and relative projective $k[G]$-modules, Valentini [25] showed for unramified covers that Tamagawa's decomposition is equivalent to cyclicity of the p-Sylow subgroups of G.