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Genus fields of congruence function fields

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Let k be a rational congruence function field and consider a finite separable extension K/k. We consider the extension K/k satisfying the following condition. For each prime in k at least one prime in K above it is tamely ramified. Then, except for constants, we find the genus field $K_{\mathfrak{ge}}$ of K/k. In general, we describe the genus field of a global function field. We present some applications and examples of the main results.

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1. Introduction and notation

C.F. Gauss [9] was the first to consider what now is called the *genus field* of a quadratic number field. H. Hasse [10] introduced genus theory for quadratic number fields describing the theory invented by Gauss by means of class field theory. H.W. Leopoldt [13] determined the genus field $K_{\mathfrak{ge}}$ of an absolute abelian number field K generalizing the work of Hasse. Leopoldt developed the theory using Dirichlet characters and relating them to the arithmetic of K. A. Fröhlich [6,7] introduced the concept of genus fields for nonabelian number fields. Fröhlich defined the genus field $K_{\mathfrak{ge}}$ of an arbitrary finite number field $K/\mathbb{Q}$ as $K_{\mathfrak{ge}} := Kk^*$ where k^* is the maximal abelian number field such that Kk^*/K is unramified. We have that k^* is the maximal abelian number field contained in $K_{\mathfrak{ge}}$. The degree $[K_{\mathfrak{ge}} : K]$ is called the *genus number* of K and the Galois group $\mathrm{Gal}(K_{\mathfrak{ge}}/K)$ is called the *genus group* of K.

If K_H denotes the Hilbert class field (HCF) of K, we have $K \subseteq K_{\mathfrak{ge}} \subseteq K_H$ and $\mathrm{Gal}(K_H/K)$ is isomorphic to the class group Cl_K of K. Then $K_{\mathfrak{ge}}$ corresponds to a subgroup G_K of Cl_K, that is, $\mathrm{Gal}(K_{\mathfrak{ge}}/K) \cong Cl_K/G_K$. The subgroup G_K is called the *principal genus* of K and $|Cl_K/G_K|$ is equal to the genus number of K.

M. Ishida [12] described the genus field $K_{\mathfrak{ge}}$ of any finite extension K of $\mathbb{Q}$, allowing ramification at the infinite primes in the extension $K_{\mathfrak{ge}}/K$. Given a number field K, Ishida found an abelian number field k_1^* and described another number field k_2^* such that $k^* = k_1^* k_2^*$ and $k_1^* \cap k_2^* = \mathbb{Q}$. The field k_1^* was constructed by means of the finite primes p such that at least one ramification index of the decomposition of p in K is not divisible by p. In other words, by those primes p such that at least one prime in K above p is tamely ramified.

We are interested in genus fields in the context of congruence (global) function fields. In this case there is no proper notion of Hilbert class field since all the constant field extensions are abelian and unramified. In fact, let K be a congruence function field with class number h_K. Then there are exactly $h := h_K$ abelian extensions $K_1, \ldots, K_h$ of K such that K_i/K are maximal unramified with exact field of constants of each K_i the same as the one of K, $\mathbb{F}_q$, the finite field of q elements and $\mathrm{Gal}(K_i/K) \cong Cl_{K,0}$ the group of classes of divisors of degree zero [2, Chapter 8, page 79].

M. Rosen [17] gave a definition of Hilbert class field of K, fixing a nonempty finite set S_∞ of prime divisors of K. Using Rosen's definition of HCF it is possible to give a proper concept of genus field along the lines of number fields. In the literature there have been different definitions of genus fields according to different HCF definitions. R. Clement [5] found a narrow genus field of a cyclic extension K of $k = \mathbb{F}_q(T)$ of prime degree dividing $q - 1$. She used a concept of HCF similar to that of a quadratic number field K. Namely, the finite abelian extension of K such that the prime ideals of the ring of integers $\mathcal{O}_K$ of K splitting fully are precisely the principal ideals generated by an element of positive norm. S. Bae and J.K. Koo [3] were able to generalize the results of Clement with the methods developed by Fröhlich [8]. They defined the genus field for general global function fields and developed the analogue of the classical genus theory.