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On the next-to-minimal weight of affine cartesian codes

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ABSTRACT

In this paper we determine many values of the second least weight of codewords, also known as the next-to-minimal Hamming weight, for a type of affine variety codes, obtained by evaluating polynomials of degree up to d on the points of a cartesian product of n subsets of a finite field $\mathbb{F}_q$. Such codes firstly appeared in a work by O. Geil and C. Thomsen (see [12]) as a special case of the so-called weighted Reed–Muller codes, and later appeared independently in a work by H. López, C. Rentería-Marquez and R. Villarreal (see [16]) named as affine cartesian codes. Our work extends, to affine cartesian codes, the results obtained by Rolland in [17] for generalized Reed–Muller codes.

1. Introduction

Let $K := \mathbb{F}_q$ be a field with q elements and let $A_1, \ldots, A_n$ be a collection of non-empty subsets of K. Consider an *affine cartesian set*

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$$\mathcal{X} := A_1 \times \cdots \times A_n := \{(a_1 : \cdots : a_n) | \, a_i \in A_i \text{ for all } i\} \subset \mathbb{A}^n,$$

where $\mathbb{A}^n$ is the n-dimensional affine space defined over K.

For a nonnegative integer d write $K[\boldsymbol{X}]_{\leq d}$ for the K-vector space formed by the polynomials in $K[X_1, \ldots, X_n]$ of degree up to d together with the zero polynomial. We denote by d_i the cardinality of A_i, for $i = 1, \ldots, n$. Clearly $|\mathcal{X}| = \Pi_{i=1}^n d_i =: \widetilde{m}$ and let $P_1, \ldots, P_{\widetilde{m}}$ be the points of $\mathcal{X}$. Define $\phi_d : K[\boldsymbol{X}]_{\leq d} \to K^{\widetilde{m}}$ as the evaluation morphism $\phi_d(g) = (g(P_1), \ldots, g(P_{\widetilde{m}}))$.

Definition 1.1. The image $C_{\mathcal{X}}(d)$ of ϕ_d is a vector subspace of $K^{\widetilde{m}}$ called the *affine cartesian code* (of order d) defined over the sets $A_1, \ldots, A_n$.

In the special case where $A_1 = \cdots = A_n = K$ we have the well-known generalized Reed–Muller code of order d. An affine cartesian code is a type of affine variety code, as defined in [10]. These codes firstly appeared in a work by O. Geil and C. Thomsen (see [12]) as a special case of the so-called weighted Reed–Muller codes, and later appeared independently in a work by H. López, C. Rentería-Marquez and R. Villarreal (see [16]) named as affine cartesian codes. In [16] the authors prove that we may ignore, in the cartesian product, sets with just one element and moreover may always assume that $2 \leq d_1 \leq \cdots \leq d_n$. The minimum distance of these codes was found independently by Geil et al. (see [12]) and López et al. (see [16]) and a formula for their dimension was presented in [16].

Theorem 1.2. *[12, Prop. 5 (with $w = 1$)] and [16, Thm. 3.1 and Thm. 3.8]*

1) The dimension of $C_{\mathcal{X}}(d)$ is $\widetilde{m}$ (i.e. ϕ_d is surjective) if $d \geq \sum_{i=1}^n (d_1 - 1)$, and for $0 \leq d < \sum_{i=1}^n (d_1 - 1)$ we have

$$\dim(C_{\mathcal{X}}(d)) = \binom{n+d}{d} - \sum_{i=1}^{n} \binom{n+d-d_i}{d-d_i} + \cdots +$$
$$(-1)^j \sum_{1 \leq i_1 < \cdots < i_j \leq n} \binom{n+d-d_{i_1} - \cdots - d_{i_j}}{d-d_{i_1} - \cdots - d_{i_j}} + \cdots + (-1)^n \binom{n+d-d_1-\cdots-d_n}{d-d_1-\cdots-d_n}$$

where we set $\binom{a}{b} = 0$ if $b < 0$.

2) The minimum distance $\delta_{\mathcal{X}}(d)$ of $C_{\mathcal{X}}(d)$ is 1, if $d \geq \sum_{i=1}^n (d_i - 1)$, and for $0 \leq d < \sum_{i=1}^n (d_i - 1)$ we have

$$\delta_{\mathcal{X}}(d) = (d_{k+1} - \ell) \prod_{i=k+2}^{n} d_i$$

where k and ℓ are uniquely defined by $d = \sum_{i=1}^k (d_i - 1) + \ell$ with $0 \leq \ell < d_{k+1} - 1$ (if $k + 1 = n$ we understand that $\prod_{i=k+2}^n d_i = 1$, and if $d < d_1 - 1$ then we set $k = 0$ and $\ell = d$).