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Subsets of $\mathbb{F}_q[x]$ free of 3-term geometric progressions ☆

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A B S T R A C T

Several recent papers have considered the Ramsey-theoretic problem of how large a subset of integers can be without containing any 3-term geometric progressions. This problem has also recently been generalized to number fields, determining bounds on the greatest possible density of ideals avoiding geometric progressions. We study the analogous problem over $\mathbb{F}_q[x]$, first constructing a set greedily which avoids these progressions and calculating its density, and then considering bounds on the upper density of subsets of $\mathbb{F}_q[x]$ which avoid 3-term geometric progressions. This new setting gives us a parameter q to vary and study how our bounds converge to 1 as it changes, and positive characteristic introduces some ex-

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Geometric progressions
Upper density

tra combinatorial structure that increases the tractability of common questions in this area.

Contents

1. Introduction

In a 1961 paper Rankin [7] introduced the idea of considering how large a set of integers can be without containing terms which are in geometric progression. He constructed a subset of the integers which includes a majority of the integers while avoiding 3-term geometric progressions. Brown and Gordon [3] noted that the set Rankin considered was the set obtained by greedily including integers subject to the condition that such integers do not create a progression involving integers already included in the set.

Other authors, including Riddell [8], Biegelböck, Bergelson, Hindman and Strauss [1], Nathanson and O'Bryant [5], and McNew [4], have refined bounds for the upper density of a set which avoids geometric progressions. Best, Huan, McNew, Miller, Powell, Tor and Weinstein [2] generalized the problem to quadratic number fields. Using many of the techniques from these other works, they obtained similar results for the density of the ideals in the ring of integers which similarly avoid geometric progressions.

The purpose of [2] was to see how the results differed when considering subsets of number fields rather than the integers. Here we investigate what happens over function fields of positive characteristic. In particular, using combinatorial tools as well as the methods of Rankin, McNew, and Best et al., we consider the size of the largest subset of the polynomial ring $\mathbb{F}_q[x]$ which avoids geometric progressions whose common ratio is a non-unit polynomial in this ring.

Remark 1.1. It is worth remarking on the choice of problem. In the integer case it is interesting to study sets which avoid 3-term geometric progressions with integral ratio as these sets have a very different flavor from those sets (called primitive sets) which avoid 2-term progressions with integral ratio. The situation is richer over $\mathbb{F}_q[x]$, for example we