Finite Fields and Their Applications 46 (2017) 179–216

Contents lists available at ScienceDirect

Finite Fields and Their Applications

www.elsevier.com/locate/ffa

Supercongruences involving dual sequences ☆

Zhi-Wei Sun

Department of Mathematics, Nanjing University, Nanjing 210093, People's Republic of China

A R T I C L E   I N F O

Article history:
Received 8 July 2016
Received in revised form 20 March 2017
Accepted 23 March 2017
Available online 19 April 2017
Communicated by Peter Jau-Shyong Shiue

MSC:
primary 11A07, 11B65
secondary 05A10, 05A19, 11B68, 11B75

Keywords:
p-adic congruences
Binomial coefficients
Combinatorial identities
Dual sequences

A B S T R A C T

In this paper we study some sophisticated supercongruences involving dual sequences. For $n = 0, 1, 2, \ldots$ define

$$d_n(x) = \sum_{k=0}^{n} \binom{n}{k}\binom{x}{k} 2^k$$

and

$$s_n(x) = \sum_{k=0}^{n} \binom{n}{k}\binom{x}{k}\binom{x+k}{k} = \sum_{k=0}^{n} \binom{n}{k}(-1)^k\binom{x}{k}\binom{-1-x}{k}.$$

For any odd prime p and p-adic integer x, we determine $\sum_{k=0}^{p-1}(\pm 1)^k d_k(x)^2$ and $\sum_{k=0}^{p-1}(2k+1)d_k(x)^2$ modulo p^2; for example, we establish the new p-adic congruence

$$\sum_{k=0}^{p-1}(-1)^k d_k(x)^2 \equiv (-1)^{\langle x\rangle_p} \pmod{p^2},$$

where $\langle x\rangle_p$ denotes the least nonnegative integer r with $x \equiv r \pmod{p}$. For any prime $p > 3$ and p-adic integer x, we determine $\sum_{k=0}^{p-1} s_k(x)^2$ modulo p^2 (or p^3 if $x \in \{0, \ldots, p-1\}$), and show that

$$\sum_{k=0}^{p-1}(2k+1)s_k(x)^2 \equiv 0 \pmod{p^2}.$$

☆ The work is supported by the National Natural Science Foundation of China (grant 11571162) and the initial version was posted to arXiv in 2015 with the ID arXiv:1512.00712.

E-mail address: zwsun@nju.edu.cn.

http://dx.doi.org/10.1016/j.ffa.2017.03.007

We also pose several related conjectures.

1. Introduction

For a sequence of numbers $a_0, a_1, a_2, \ldots$, its *dual sequence* $a_0^*, a_1^*, a_2^*, \ldots$ is given by

$$a_n^* = \sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k \quad (n = 0, 1, 2, \ldots). \tag{1.1}$$

It is well-known that $(a_n^*)^* = a_n$ for all $n = 0, 1, 2, \ldots$. One may consult [15] for some combinatorial identities involving dual sequences. The author [21, Theorem 2.2] showed that if p is an odd prime, $a_0, a_1, \ldots, a_{p-1}$ are p-adic integers and m is an integer with $p \nmid m(m-4)$ then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} a_k^*}{(4-m)^k} \equiv \left(\frac{m(m-4)}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k} a_k}{m^k} \pmod{p},$$

where $(\frac{\cdot}{p})$ denotes the Legendre symbol.

Let p be any odd prime. There are various interesting p-adic congruences related to finite fields, see, e.g., [1,3,7,22,24]. The author and R. Tauraso [23, (1.9)] showed that

$$\sum_{k=0}^{p-1} \binom{2k}{k} \equiv \left(\frac{p}{3}\right) \pmod{p^2},$$

where $(-)$ is the Legendre symbol. In [17] the author determined $\sum_{k=0}^{p-1} \binom{2k}{k}/m^k$ modulo p^2 for any integer $m \not\equiv 0 \pmod{p}$, and moreover he proved that

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{2^k} \equiv \left(\frac{-1}{p}\right) - p^2 E_{p-3} \pmod{p^3},$$

where $E_0, E_1, E_2, \ldots$ are the Euler numbers given by

$$E_0 = 1 \quad \text{and} \quad \sum_{\substack{k=0 \\ 2 \mid k}}^{n} \binom{n}{k} E_{n-k} = 0 \quad \text{for } n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}.$$

If $a_n = \binom{2n}{n} = \binom{-1/2}{n}(-4)^n$ for $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$, then we have $a_n^* = (-1)^n T_n$ for all $n \in \mathbb{N}$ by [4, (3.86)], where the central trinomial coefficient T_n is the constant term of $(1 + x + x^{-1})^n$. In [20] the author showed that