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Torsion-free rank one sheaves over del Pezzo orders ☆

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Let $\mathcal{A}$ be a del Pezzo order on the projective plane over the field of complex numbers. We prove that every torsion-free $\mathcal{A}$-module of rank one can be deformed into a locally free $\mathcal{A}$-module of rank one.

Introduction

An *order* on an algebraic variety X is a torsion-free coherent sheaf of $\mathcal{O}_X$-algebras whose generic stalk is a central division algebra over the function field of X. A surface

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together with an order on it can be thought of as a noncommutative surface. In this article we are interested in terminal del Pezzo orders on the projective plane $\mathbb{P}^2$ over the field of complex numbers $\mathbb{C}$. These orders are noncommutative analogues of classical del Pezzo surfaces and have been completely classified by D. Chan and C. Ingalls in the course of their proof of the minimal model program for orders over surfaces, see [3].

Let $\mathcal{A}$ be a terminal del Pezzo order on $\mathbb{P}^2$. Left $\mathcal{A}$-modules which are locally free and generically of rank one can be thought of as line bundles on this noncommutative surface. There is a quasi-projective coarse moduli scheme for these line bundles [6], a noncommutative analogue of the classical Picard scheme. To compactify this moduli scheme, that is to get a projective moduli scheme, one has to allow torsion-free left $\mathcal{A}$-modules which are generically of rank one.

We study the boundary of this compactification by studying the deformation theory of torsion-free $\mathcal{A}$-modules. The main result of this article is the following

Theorem. *Let $\mathcal{A} \neq \mathcal{O}_{\mathbb{P}^2}$ be a terminal del Pezzo order on $\mathbb{P}^2$ over $\mathbb{C}$. Then every torsion-free $\mathcal{A}$-module E of rank one can be deformed to a locally free $\mathcal{A}$-module E'.*

As a corollary, we obtain that every irreducible component of the compactification of the noncommutative Picard scheme contains a point defined by an $\mathcal{A}$-line bundle.

The structure of this paper is as follows. We review the definition and some basic facts about terminal del Pezzo orders in section 1. In section 2 we study in detail the local deformation theory of $\mathcal{A}$-modules in this setting. We look at the homological algebra of torsion-free $\mathcal{A}$-modules and study the compactification of the noncommutative Picard scheme and some of its properties in section 3. In the final section 4 we study the global deformation theory and prove the main result.

1. Noncommutative del Pezzo surfaces

Let X be a smooth projective surface over $\mathbb{C}$.

Definition 1.1. An *order* $\mathcal{A}$ on X is sheaf of associative $\mathcal{O}_X$-algebras such that

- $\mathcal{A}$ is coherent and torsion-free as an $\mathcal{O}_X$-module, and
- the stalk $\mathcal{A}_\eta$ at the generic point $\eta \in X$ is a central division algebra over the function field $\mathbb{C}(X) = \mathcal{O}_{X,\eta}$ of X.

We can now look at all orders in $\mathcal{A}_\eta$ and order them by inclusion. A maximal element will be called a maximal order. These are the algebras we are interested in. Maximal orders have some nice properties, for example they are locally free $\mathcal{O}_X$-modules.

Furthermore, it is well known that there is a largest open subset $U \subset X$ on which $\mathcal{A}$ is even an Azumaya algebra, see for example [9, Proposition 6.2]. The complement