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# The involution width of finite simple groups

Alexander J. Malcolm

*School of Mathematics, University of Bristol, Bristol BS8 1TW, UK*

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ABSTRACT

For a finite group generated by involutions, the involution width is defined to be the minimal $k \in \mathbb{N}$ such that any group element can be written as a product of at most $k$ involutions. We show that the involution width of every non-abelian finite simple group is at most 4. This result is sharp, as there are families with involution width precisely 4.



## 1. Introduction

Let $G$ be a finite group generated by involutions. The involution width, denoted $\mathrm{iw}(G)$, is defined to be the minimal $k \in \mathbb{N}$ such that any element of $G$ can be written as a product of at most $k$ involutions. It is well known that all non-abelian finite simple groups are generated by their involutions. Furthermore, it was proved by Liebeck and Shalev ([36, 1.4]) that there exists an absolute constant $N$ that bounds the involution width of all finite simple groups. The purpose of this paper is to obtain the minimal value for $N$.

*E-mail address:* alex.malcolm@bristol.ac.uk.

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The involution width was first considered in the case of special linear groups by Wonenburger [52], and later by Gustafson et al. [26] and Knüppel and Nielsen [29]. Further work has been completed on orthogonal groups (Knüppel and Thomsen, [30]) and the exceptional group $F_4(K)$ (Austin, [2]).

More recently, various problems have been resolved involving the related notions of real or strongly real groups. An element $x \in G$ is real if $x^g = x^{-1}$ for some $g \in G$. Furthermore $x$ is strongly real if $g$ can be taken to be an involution. We call the group $G$ (strongly) real if all of its elements are (strongly) real. It follows easily that $x \in G$ is strongly real if and only if $x$ is a product of 2 involutions and so the strongly real groups are precisely those of involution width at most 2. The classification of strongly real, finite simple groups was completed in 2010 after work by a number of authors (see [21,28,42,47,17] and Theorem 2.1 below). The work of this paper addresses the remaining finite simple groups and is summarised by the following theorem.

**Theorem 1.** *Every non-abelian finite simple group has involution width at most 4.*

Note that the upper bound 4 is sharp, as certain families (for example $PSL_n(q)$ such that $n, q \geq 6$ and $\gcd(n, q-1) = 1$) do have involution width 4 (see Theorem 3.6).

The involution width is one of a number of width questions that have been considered about simple groups in recent literature. For example, [32] settles the longstanding conjecture of Ore that the commutator width of any finite non-abelian simple group $G$ is exactly 1. Also, $G$ is generated by its set of squares and the width in this case is 2 [33]. More generally, given any two non-trivial words $w_1, w_2$, if $G$ is of large enough order then $w_1(G)w_2(G) \supseteq G \backslash \{1\}$ [25].

Here are some remarks on the proof of Theorem 1. For alternating groups we find the involution width directly by studying the disjoint cycle decomposition of elements. For groups $G$ of Lie type we adopt a different approach: we aim to find particular regular semisimple elements $x, y \in G$ such that $x$ and $y$ are strongly real and $G \backslash \{1\} \subseteq x^G y^G$. It then follows that every element is a product of at most 4 involutions. To do this we make extensive use of the character theory of finite groups of Lie type, building on methods first seen in [38] and [24]. Substantial difficulties are faced in the case of unitary groups, where we develop the theory of minimal degree characters using dual pairs (Sec. 4). Similarly problematic are a number of exceptional groups of Lie type and in these instances we use an inductive approach, restricting to subgroups of $G$ for which the involution width is known (Sec. 5).

Naturally the involution width problem can be generalised to elements of order $p$, for any fixed prime $p$. This has been resolved in the case of alternating groups and work on the simple groups of Lie type will be forthcoming.

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