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# Involutions and stable subalgebras

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Given a central simple algebra with involution over an arbitrary field, étale subalgebras contained in the space of symmetric elements are investigated. The method emphasizes the similarities between the various types of involutions and privileges a unified treatment for all characteristics whenever possible. As a consequence a conceptual proof of a theorem of Rowen is obtained, which asserts that every division algebra of exponent two and degree eight contains a maximal subfield that is a triquadratic extension of the centre.



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## 1. Introduction

We investigate étale algebras in the space of symmetric elements of a central simple algebra with involution over an arbitrary field, emphasizing the similarities between the various types of involutions and avoiding restrictions on the characteristic. In Section 2 and Section 3 we recall the terminology and some crucial techniques for algebras with involution. We enhance this terminology in a way that allows us to avoid unnecessary case distinctions in the sequel, according to the different types of involution and to the characteristic. To this end we introduce in Section 3 the notion of *capacity* of an algebra with involution. It is defined to be the degree of the algebra if the involution is orthogonal or unitary, and half the degree if the involution is symplectic. In Section 5 we isolate a notion of *neat* subalgebra, which captures the features of separable field extensions of the centre consisting of symmetric elements while avoiding the pathologies that may arise with arbitrary étale algebras. We prove their existence and determine their maximal dimension to be equal to the capacity (Theorem 4.1 and Proposition 5.6). In Section 6, given a neat quadratic subalgebra $K$, we establish the existence of a neat subalgebra $L$ linearly disjoint from $K$ and centralising $K$ and such that the composite $KL$ is a neat algebra of maximal dimension (Theorem 6.10). In Section 7 we apply this result to construct neat biquadratic subalgebras in the space of symmetric elements of central simple algebras of degree 4 with orthogonal or unitary involutions, and similarly of central simple algebras of degree 8 with symplectic involutions (Theorem 7.4). As a consequence, we obtain a conceptual proof of a theorem of Rowen, which asserts that division algebras of exponent 2 and degree 8 are elementary abelian crossed products, i.e., they contain a maximal subfield which is a triquadratic separable extension of the centre (Corollary 7.7). Actually we obtain directly a refined version of this result which says that any symplectic involution on a central simple algebra of degree 8 stabilizes some triquadratic étale extension of the centre (Theorem 7.6). This has been proven in [7, Lemma 6.1] for division algebras in characteristic different from two, but there the proof uses Rowen's Theorem, which we obtain here as a consequence. This illustrates the usefulness of involutions in the investigation of central simple algebras of exponent two.

The results of this paper will be used in [4], which proposes a common approach to the definition of the first cohomological invariant (discriminant) of the involutions of capacity four of various types through Pfister forms in arbitrary characteristic.

## 2. Algebras

In this preliminary section we introduce and recall some definitions and facts from the theory of finite-dimensional simple and semisimple algebras. Our standard references are [11] and [8].

Let $F$ be an arbitrary field. For a commutative $F$-algebra $K$ we set $[K : F] = \dim_F K$. Recall that an $F$-algebra is *étale* if it is isomorphic to a finite product of finite separable