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Local Bezout estimates and multiplicities of parameter and primary ideals ☆

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Let $\mathfrak{q}$ denote an $\mathfrak{m}$-primary ideal of a d-dimensional local ring $(A, \mathfrak{m})$. Let $\underline{a} = a_1, \dots, a_d \subset \mathfrak{q}$ be a system of parameters. Then there is the following inequality for the multiplicities $c \cdot e(\mathfrak{q}; A) \le e(\underline{a}; A)$ where c denotes the product of the initial degrees of a_i in the form ring $G_A(\mathfrak{q})$. The aim of the paper is a characterization of the equality as well as a description of the difference by various homological methods via Koszul homology. To this end we have to characterize when the sequence of initial elements $\underline{a^\star} = a_1^\star, \dots, a_d^\star$ is a homogeneous system of parameters of $G_A(\mathfrak{q})$. In the case of $\dim A = 2$ this leads to results on the local Bezout inequality. In particular, we give several equations for improving the classical Bezout inequality to an equality.

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1. Introduction

Let $C, D \subset \mathbb{A}_k^2$ be two affine plane curves with no components in common. Let $f, g \in k[x, y]$ denote their defining equations, i.e. $C = V(f)$ and $D = V(g)$. Suppose that $0 \in C \cap D$. Let $A = k[x, y]_{(x,y)}$ denote the local ring at the origin. Then the local Bezout inequality in the plane says that

$$e(f, g; A) \geq c \cdot d,$$

where $e(f, g; A)$ denotes the local intersection multiplicity and c and d denote the (initial) degrees of C and D respectively. Equality holds if and only if C and D intersect transversally in 0, i.e. if and only if the initial forms $f^\star, g^\star \in k[X, Y]$ form a homogeneous system of parameters in $k[X, Y]$. This is a classical result, see [3] or [6] for references.

One of the aims of the present paper is the following Theorem.

Theorem 1.1. *With the previous notation there are the following results:*

(a) $e(f, g; A) = c \cdot d + t + \ell$, *where t denotes the number of common tangents in* $(0, 0)$ *counted with multiplicities and ℓ is a non-negative number defined in local data.*
(b) $e(f, g; A) = c \cdot d + e(f_1, g_1; A[x/y]) + e(f_2, g_2; A[y/x]) - e(f_1, g_1; A[x/y, y/x])$, *where $f_i, g_i, i = 1, 2$, denote the corresponding strict transforms of f, g in the blowing up rings $A[x/y]$ and $A[y/x]$.*
(c) *Suppose C and D do not intersect transversally in the origin. Then*

$$e(f, g; A) \leq c \cdot d + e(f_1, g_1; A[x/y]) + e(f_2, g_2; A[y/x])$$

with equality if and only if one of the coordinate axes is a common tangent in $(0, 0)$.

For the precise notion of e we refer to Remark 8.2 (B). The proof of Theorem 1.1 is given in Theorems 7.1, 9.3 and 10.4. The inequality $e(f, g; A) \geq c \cdot d + t$ was proved by Bydžovský (see [4]) through the study of resultants. His result was one of the motivations for the investigations in the present paper. The formula in Theorem 1.1 (b) was inspired by those of Greuel, Lossen and Shustin (see [8, Proposition 3.21]). In fact, we correct their formula by showing that it depends upon the embedding $C, D \subset \mathbb{A}_k^2$ in contrast to the claim in the proof of [8, Proposition 3.21].

Another motivation for the authors was the paper [11]. Let $\underline{a} = a_1, \ldots, a_d$ be a system of parameters in the local ring $(A, \mathfrak{m})$. Let $\mathfrak{q}$ denote an $\mathfrak{m}$-primary ideal with $(\underline{a}) \subseteq \mathfrak{q}$ and $a_i \in \mathfrak{q}^{c_i} \setminus \mathfrak{q}^{c_i+1}, i = 1, \ldots, d$. Then $c_1 \cdots c_d \; e(\mathfrak{q}; A) \leq e(\underline{a}; A)$ (see Lemma 3.1). In the case of $\mathfrak{q} = \mathfrak{m}$ it was claimed in [11] that equality holds if and only if the sequence of initial elements $a_1^\star, \ldots, a_d^\star$ forms an $G_A(\mathfrak{q})$-regular sequence. This is not true (see the Examples 3.2). Therefore we investigate the relation between both of these multiplicities.

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