Journal of Algebra 488 (2017) 145–200

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

On universal quadratic identities for minors of quantum matrices

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A R T I C L E I N F O

Article history:
Received 15 December 2016
Available online 17 June 2017
Communicated by Masaki Kashiwara

MSC:
16T99
05C75
05E99

Keywords:
Quantum matrix
Quadratic identity for minors
Planar graph
Cauchon diagram
Path matrix
Lindström's Lemma

A B S T R A C T

We give a complete combinatorial characterization of homogeneous quadratic relations of "universal character" valid for minors of quantum matrices (more precisely, for minors in the quantized coordinate ring $\mathcal{O}_q(\mathcal{M}_{m,n}(\mathbb{K}))$ of $m \times n$ matrices over a field $\mathbb{K}$, where $q \in \mathbb{K}^*$). This is obtained as a consequence of a study of quantized minors of matrices generated by paths in certain planar graphs, called *SE-graphs*, generalizing the ones associated with Cauchon diagrams. Our efficient method of verifying universal quadratic identities for minors of quantum matrices is illustrated with many appealing examples.

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http://dx.doi.org/10.1016/j.jalgebra.2017.05.034
0021-8693/

1. Introduction

The idea of quantization has proved its importance to bridge the commutative and noncommutative versions of certain algebraic structures and promote better understanding various aspects of the latter versions. One popular structure studied for the last three decades (as an important part of the study of algebraic quantum groups) is the quantized coordinate ring $\mathcal{R} = \mathcal{O}_q(\mathcal{M}_{m,n}(\mathbb{K}))$ of $m \times n$ matrices over a field $\mathbb{K}$, where q is a nonzero element of $\mathbb{K}$; it is usually called the *algebra of* $m \times n$ *quantum matrices*. Here $\mathcal{R}$ is the $\mathbb{K}$-algebra generated by the entries (indeterminates) of an $m \times n$ matrix X subject to the following (quasi)commutation relations due to Manin [12]: for $1 \le i < \ell \le m$ and $1 \le j < k \le n$,

$$x_{ij}x_{ik} = qx_{ik}x_{ij}, \qquad x_{ij}x_{\ell j} = qx_{\ell j}x_{ij}, \tag{1.1}$$
$$x_{ik}x_{\ell j} = x_{\ell j}x_{ik} \quad \text{and} \quad x_{ij}x_{\ell k} - x_{\ell k}x_{ij} = (q - q^{-1})x_{ik}x_{\ell j}.$$

This paper is devoted to quadratic identities for minors of quantum matrices (usually called quantum minors or quantized minors or q-minors). For representative cases, aspects and applications of such identities, see, e.g., [6–10,14,15] (where the list is incomplete). We present a novel, and rather transparent, combinatorial method which enables us to completely characterize and efficiently verify homogeneous quadratic identities of universal character that are valid for quantum minors.

The identities of our interest can be written as

$$\sum (s_i q^{\delta_i} [I_i|J_i]_q \, [I'_i|J'_i]_q \colon i = 1, \dots, N) = 0, \tag{1.2}$$

where $\delta_i \in \mathbb{Z}$, $s_i \in \{+1, -1\}$, and $[I|J]_q$ denotes the quantum minor whose rows and columns are indexed by $I \subseteq [m]$ and $J \subseteq [n]$, respectively. (Hereinafter, for a positive integer n', we write $[n']$ for $\{1, 2, \dots, n'\}$.) The homogeneity means that each of the sets $I_i \cup I'_i$, $I_i \cap I'_i$, $J_i \cup J'_i$, $J_i \cap J'_i$ does not depend i, and the term "universal" means that (1.2) should be valid independently of $\mathbb{K}, q$ and a q-matrix (a matrix whose entries obey Manin's relations and, possibly, additional ones). Note that any quadruple $(I|J, I'|J')$, referred to as a *cortege* later on, may be repeated in (1.2) several times.

Our approach is based on two sources. The first one is the *flow-matching method* elaborated in [4] to characterize quadratic identities for usual minors (viz. for $q = 1$). In that case the identities are viewed simpler than (1.2), namely, as

$$\sum (s_i [I_i|J_i] \, [I'_i|J'_i] \colon i = 1, \dots, N) = 0. \tag{1.3}$$

(In fact, [4] deals with natural analogs of (1.3) over commutative semirings, e.g. the tropical semiring $(\mathbb{R}, +, \max)$.) In the method of [4], each cortege $S = (I|J, I'|J')$ is associated with a certain set $\mathcal{M}(S)$ of *feasible matchings* on the set $(I \triangle I') \sqcup (J \triangle J')$ (where $A \triangle B$ denotes the symmetric difference $(A - B) \cup (B - A)$, and $A \sqcup B$ the disjoint