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Finitely supported $*$-simple complete ideals and multiplicities in a regular local ring

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Let $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ be regular local rings of $\dim(S) = \dim(R) \geq 2$ such that S birationally dominates R, and let $\mathcal{V}$ be the order valuation ring of S with corresponding valuation $\nu := \mathrm{ord}_S$. Assume that $I^S \neq S$ and $\nu \in \mathrm{Rees}_S\, I^S$. Let $u := \alpha t$ with $IS = \alpha I^S$, where $\alpha \in S$. Then $\mathcal{V} = \mathcal{W} \cap \mathcal{Q}(R)$ with $\mathcal{W} = (\overline{R[It]})_Q = (\overline{S[I^S u]})_{Q'}$, where $Q \in \mathrm{Min}(\mathfrak{m}\overline{R[It]})$ and $Q' \in \mathrm{Min}(\mathfrak{n}\overline{S[I^S u]})$. Let P, P' be the center of $\mathcal{W}$ on $R[It]$ and $S[I^s u]$, respectively. We prove that if $[\frac{S}{\mathfrak{n}} : \frac{R}{\mathfrak{m}}] = 1$, then $\frac{R[It]}{P} = \frac{S[I^s u]}{P'}$. Let I be a finitely supported complete $\mathfrak{m}$-primary ideal of a regular local ring $(R, \mathfrak{m})$ of dimension $d \geq 2$. Let T be a terminal base point of I and V be the $\mathfrak{m}_T$-adic order valuation of T with corresponding valuation $v := \mathrm{ord}_T$. Let $n \geq 1$ be an integer. Assume that $I^T = \mathfrak{m}_T^n$ and $[\frac{T}{\mathfrak{m}_T} : \frac{R}{\mathfrak{m}}] = 1$. Let $P \in \mathrm{Min}(\mathfrak{m}R[It])$ such that $P = Q \cap R[It]$ with $V = (\overline{R[It]})_Q \cap \mathcal{Q}(R)$, where $Q \in \mathrm{Min}(\mathfrak{m}\overline{R[It]})$. We prove that the quotient ring $\frac{R[It]}{P}$ is d-dimensional normal Cohen–Macaulay standard graded domain over k with the multiplicity n^{d-1}. In particular, $\frac{R[It]}{P}$ is regular if and only if $n = 1$. We prove that $k := \frac{R}{\mathfrak{m}}$ is relatively algebraically closed in $k_v := \frac{V}{\mathfrak{m}_V}$. Also we determine the multiplicity of $\frac{R[It]}{P}$, and

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we prove that if $I^T = \mathfrak{m}_T$, then $\frac{R[It]}{P}$ is regular if and only if $[\frac{T}{\mathfrak{m}_T} : \frac{R}{\mathfrak{m}}] = 1$.

1. Introduction

All rings we consider are assumed to be commutative with an identity element. We use the concept of complete ideals as defined and discussed in Swanson–Huneke [20, Chapters 5, 6, 14]. We also use a number of concepts considered in Lipman's paper [16]. Let $(R, \mathfrak{m})$ be a regular local ring of dimension $d \geq 2$. Lipman considers the structure of a certain class of complete ideals of R, the finitely supported complete ideals, in [16]. He proves a factorization theorem for the finitely supported complete ideals that extends the factorization theory of complete ideals in a two-dimensional regular local ring as developed by Zariski [22, Appendix 5]. The product of two complete ideals in a two-dimensional regular local ring is again complete. This no longer holds in higher dimension, [3] or [13]. To consider the higher dimensional case, one defines for ideals I and J the $*$-product, $I * J$ to be the completion of IJ. A complete ideal I in a commutative ring R is said to be **$*$-simple** if $I \neq R$ and if $I = J * L$ with ideals J and L in R implies that either $J = R$ or $L = R$.

Another concept used by Zariski in [22] is that of the transform of an ideal; the complete transform of an ideal is used in [16] and [5].

Definition 1.1. Let $R \subseteq T$ be unique factorization domains (UFDs) with R and T having the same field of fractions, and let I be an ideal of R not contained in any proper principal ideal.

(1) The **transform** of I in T is the ideal $I^T = a^{-1}IT$, where aT is the smallest principal ideal in T that contains IT.
(2) The **complete transform** of I in T is the completion $\overline{I^T}$ of I^T.

A proper ideal I in a commutative ring R is **simple** if $I \neq L \cdot H$, for any proper ideals L and H. An element $\alpha \in R$ is said to be **integral over** I if α satisfies an equation of the form

$$\alpha^n + r_1\alpha^{n-1} + \cdots + r_n = 0, \quad \text{where} \quad r_i \in I^i.$$

The set of all elements in R that are integral over an ideal I forms an ideal, denoted by $\overline{I}$ and called the **integral closure** of I. An ideal I is said to be **complete** (or, **integrally closed**) if $I = \overline{I}$.