Journal of Algebra 488 (2017) 315–348

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Directed unions of local quadratic transforms of a regular local ring [☆]

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ARTICLE INFO

Article history:
Received 24 February 2014
Available online 27 June 2017
Communicated by Bernd Ulrich

MSC:
primary 13A30, 13C05
secondary 13E05, 13H15

Keywords:
Local quadratic transform
Infinite directed family
Switches strongly infinitely often
Rank and rational rank of a valuation domain
Transform of an ideal
Monomial ideal
Valuation ideal

ABSTRACT

Let $(R, \mathfrak{m})$ be a d-dimensional regular local domain with $d \geq 2$ and let V be a valuation domain birationally dominating R such that the residue field of V is algebraic over $R/\mathfrak{m}$. Let v be a valuation associated to V. Associated to R and V there exists an infinite directed family $\{(R_n, \mathfrak{m}_n)\}_{n \geq 0}$ of d-dimensional regular local rings dominated by V with $R = R_0$ and R_{n+1} the local quadratic transform of R_n along V. Let $S := \bigcup_{n \geq 0} R_n$. Abhyankar proves that $S = V$ if $d = 2$. Shannon observes that often S is properly contained in V if $d \geq 3$, and Granja gives necessary and sufficient conditions for S to be equal to V. The directed family $\{(R_n, \mathfrak{m}_n)\}_{n \geq 0}$ and the integral domain $S = \bigcup_{n \geq 0} R_n$ may be defined without first prescribing a dominating valuation domain V. If $\{(R_n, \mathfrak{m}_n)\}_{n \geq 0}$ switches strongly infinitely often, then $S = V$ is a rank one valuation domain and for nonzero elements f and g in $\mathfrak{m}$, we have $\frac{v(f)}{v(g)} = \lim_{n \to \infty} \frac{\operatorname{ord}_{R_n}(f)}{\operatorname{ord}_{R_n}(g)}$. If $\{(R_n, \mathfrak{m}_n)\}_{n \geq 0}$ is a family of monomial local quadratic transforms, we give necessary and sufficient conditions for $\{(R_n, \mathfrak{m}_n)\}_{n \geq 0}$ to switch strongly infinitely often. If these conditions hold, then $S = V$ is a rank one valuation domain of

☆ Correspondence with Alan Loper, Bruce Olberding and Hans Schoutens that motivated our interest in the work of Shannon and Granja on infinite directed unions of local quadratic transformations is gratefully acknowledged.

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1 This paper was supported by Faculty Research Fund, Sungkyunkwan University, 2013.

http://dx.doi.org/10.1016/j.jalgebra.2017.06.013

rational rank d and v is a monomial valuation. Assume that V is rank one and birationally dominates S. Let $s = \sum_{i=0}^{\infty} v(\mathfrak{m}_i)$. Granja, Martinez and Rodriguez show that $s = \infty$ implies $S = V$. We prove that s is finite if V has rational rank at least 2. In the case where V has maximal rational rank, we give a sharp upper bound for s and show that s attains this bound if and only if the sequence switches strongly infinitely often.

1. Introduction

Let $R = R_0$ be a d-dimensional regular local ring, and for each integer $n \geq 0$, let R_{n+1} be a d-dimensional local quadratic transform of R_n. Thus $\{(R_n, \mathfrak{m}_n)\}_{n \geq 0}$ is a directed family of d-dimensional regular local rings. Let $S := \bigcup_{n \geq 0} R_n$. If $d = 2$, Abhyankar proves in [1] that S is always a valuation domain. In the case where $d \geq 3$, Shannon in [12] and later Granja in [3] consider conditions in order that S be a valuation domain. In this connection, Shannon gives the following definition in [12, page 314].

Definition 1.1. Let $\{(R_n, \mathfrak{m}_n)\}_{n \geq 0}$ be an infinite directed family of local quadratic transforms of a regular local ring $(R, \mathfrak{m})$. We say that $\{(R_n, \mathfrak{m}_n)\}_{n \geq 0}$ **switches strongly infinitely often** if there does not exist an integer j and a height one prime ideal $\mathfrak{p}_j$ of R_j with the property that $\bigcup_{n=0}^{\infty} R_n \subset (R_j)_{\mathfrak{p}_j}$.

Assume that V is a rank one valuation domain that birationally dominates $S := \bigcup_{n \geq 0} R_n$. If V is non-discrete, Shannon proves in [12, Proposition 4.18] that $S = V$ if and only if $\{(R_n, \mathfrak{m}_n)\}_{n \geq 0}$ switches strongly infinitely often. It is observed in [2, Theorem 6] that the proof given by Shannon also holds if V is rank one discrete.

Granja [3, Proposition 7] shows that if $S := \bigcup_{n \geq 0} R_n$ is a valuation domain V, then V has real rank either one or two, and in [3, Theorem 13], he characterizes the sequence of local quadratic transforms of R along V. If V has rank one, then $\{(R_n, \mathfrak{m}_n)\}_{n \geq 0}$ switches strongly infinitely often. If V has rank two, then the value group of V is $\mathbb{Z} \oplus G$, where G has rational rank one. In this case Granja proves [3, Theorem 13] that the sequence $\{(R_n, \mathfrak{m}_n)\}_{n \geq 0}$ is height one directed as in Definition 1.2.

Definition 1.2. Let $\{(R_n, \mathfrak{m}_n)\}_{n \geq 0}$ be an infinite directed family of local quadratic transforms of a regular local ring $(R, \mathfrak{m})$. The sequence $\{(R_n, \mathfrak{m}_n)\}_{n \geq 0}$ is **height one directed** if there exists a nonnegative integer j and a height one prime ideal $\mathfrak{p}$ of R_j such that $\bigcup_{n=0}^{\infty} R_n \subset (R_j)_{\mathfrak{p}}$, and if for some nonnegative integer k and some height one prime ideal $\mathfrak{q}$ of R_k we have $\bigcup_{n=0}^{\infty} R_n \subset (R_k)_{\mathfrak{q}}$, then $(R_j)_{\mathfrak{p}} = (R_k)_{\mathfrak{q}}$.

Let $\{(R_n, \mathfrak{m}_n)\}_{n \geq 0}$ be a directed family of Noetherian local domains. Assume that ord_{R_n} defines a valuation for each n. If $\bigcup_{n=0}^{\infty} R_n = V$ is a rank one valuation domain,