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# Irreducible representations of the plactic algebra of rank four [☆]

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Irreducible representations of the plactic monoid $M$ of rank four are studied. Certain concrete families of simple modules over the plactic algebra $K[M]$ over a field $K$ are constructed. It is shown that the Jacobson radical $J(K[M])$ of $K[M]$ is nilpotent. Moreover, the congruence $\rho$ on $M$ determined by $J(K[M])$ coincides with the intersection of the congruences determined by the primitive ideals of $K[M]$ corresponding to the constructed simple modules. In particular, $M/\rho$ is a subdirect product of the images of $M$ in the corresponding endomorphism algebras.



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Simple module
Jacobson radical

## 1. Introduction

For an integer $n \geq 1$, the finitely presented monoid $M_n = \langle a_1, \ldots, a_n \rangle$ defined by the relations

$$\begin{aligned} a_i a_k a_j &= a_k a_i a_j \qquad \text{for } i \leq j < k, \\ a_j a_i a_k &= a_j a_k a_i \qquad \text{for } i < j \leq k \end{aligned}$$

is referred to as the plactic monoid of rank $n$, [18]. The origins of this monoid come from the work of Schensted [23] and Knuth [11] concerning certain combinatorial problems and operations on Young tableaux. In particular, the elements of $M_n$ can be uniquely written in a canonical form that allows to identify them with semistandard Young tableaux. This leads to very deep applications of the plactic monoids in several areas of mathematics. In particular, they have already proved to be a classical tool in representation theory of the full linear group and in the theory of symmetric functions, via the Littlewood–Richardson rule (cf. [8,17]). They also play an important role in quantum groups (in the context of crystal bases) and in the area of classical Lie algebras, [6,16,19]. According to Schützenberger, plactic monoids should be considered among the most fundamental monoids in algebra, [24]. The combinatorics of $M_n$ has been extensively studied, [17,18], recently also in the context of Gröbner–Shirshov bases and semigroup identities (cf. [1, 2,14,15]), while the algebraic structure of the monoid algebra $K[M_n]$ of $M_n$ over a field $K$ and irreducible representations of $M_n$ are known only for $n \leq 3$, [4,13]. If $n < 3$, then $K[M_n]$ is prime and semiprimitive, and the structure of $K[M_n]$ is quite well understood. If $n = 3$ then the algebra $K[M_n]$ is not prime. It has exactly two minimal prime ideals, that are determined by a homogeneous congruence on $M_n$, but $K[M_n]$ is still semiprimitive. If additionally the base field $K$ is uncountable and algebraically closed, a complete classification of irreducible representations of $K[M_n]$ and of the corresponding primitive ideals is known. On the other hand, if $n > 3$, then $K[M_n]$ is not even semiprime and its structure seems to be much more complicated.

In view of the role played by the plactic monoid in various areas of algebra, the search for irreducible representations of $M_n$ seems a natural and challenging task. The following problem, perhaps very difficult in full generality, is one of our main motivations.

**Problem.** Determine irreducible representations of the plactic monoid $M_n$ of any finite rank $n$.

Related important questions are concerned with the radical of the plactic algebra. They are also motivated by general, very difficult, open problems on the radical of finitely presented algebras, see [25]. In particular, the following conjecture seems natural.

**Conjecture.** *The Jacobson radical of the plactic algebra $K[M_n]$ is a nilpotent ideal.*