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# The lowest two-sided cell of a Coxeter group with complete graph ☆

Xun Xie

*School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, China*

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ABSTRACT

In this paper we give a description of the structure of the based ring of the lowest two-sided cell for a weighted Coxeter group with complete graph. It is proved that this ring is generated, in a simple way, by some subrings which are isomorphic to the based rings of the lowest two-sided cells for affine Weyl groups of type $\tilde{A}_1$ or $\tilde{A}_2$.



## 1. Introduction

The cells of Weyl groups arose from the study of the primitive ideals of the enveloping algebras of semisimple Lie algebras. These cells can be defined by using Kazhdan–Lusztig basis of Hecke algebras in a combinatorial way (see [2]), and can be generalized to any weighted Coxeter group (see [3,8]). The cells play an important role in the study of representations of Hecke algebras. For example, the cells of affine Weyl groups provide a

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*E-mail address:* xieg7@163.com.

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lot of information about the representations of affine Hecke algebras (see [4–7,14]). One of the interesting things is to study the based ring defined in [5], which is also called the asymptotic Hecke algebra. For affine Weyl groups, the structure of based rings is related to representations of Hecke algebras and algebraic groups; see [7, §10.5] for a conjecture of Lusztig, and see [11–13,1,16] for some related results.

Recently, Xi proves in [15] that, for a Coxeter group with complete graph, the **a**-function is bounded and there exists a unique lowest two-sided cell, and in [9] Shi studies in detail the reduced expressions of elements of a Coxeter group with complete graph. Then it is natural to ask the following question: what is the structure of the based ring of the lowest two-sided cell for a Coxeter group with complete graph? This paper is committed to solve this problem.

To state our main results, we need some notations. Let $(W, S, L)$ be a weighted Coxeter group with complete graph. Let $K_0 = \max\{L(w_I) \mid I \subseteq S \text{ with } W_I \text{ finite}\}$. It is already known that the **a**-function of $W$ is bounded by $K_0$, see [15,10]. Let $P_0$ (resp. $\mathcal{P}_0$) be the set of $w_I \in W$ (resp. $I$) such that $L(w_I) = K_0$. By Theorem 2.1(i) and Proposition 3.4, there is a unique two-sided cell $\mathbf{c}_0$ of $W$ containing $P_0$, which is called the lowest two-sided cell of $W$. For $d \in P_0$, let $U_d$ be the set of $y \in W$ such that $l(dy) = l(d) + l(y)$, and let $B_d$ be the set of $b \in U_d^{-1}$ such that $bds \notin \mathbf{c}_0$ for any $s \in \mathcal{R}(d)$. Then $\mathbf{c}_0$ can be written as disjoint unions (see Corollary 3.9 and Theorem 3.10)

$$\mathbf{c}_0 = \bigsqcup_{d \in P_0} B_d d U_d = \bigsqcup_{d', d \in P_0} B_{d'} P_{d',d} B_d^{-1}.$$

where $P_{d',d}$ is the set of all the $w \in W$ such that $\mathcal{L}(w) = \mathcal{L}(d')$ and $\mathcal{R}(w) = \mathcal{R}(d)$.

Let $P = \bigcup_{d,d' \in P_0} P_{d',d}$. If $x, y \in P$ such that $\mathcal{R}(x) = \mathcal{L}(y) = \mathcal{L}(d)$ for some $d \in P_0$, we define the composition of $x$ and $y$ to be $x \circ y = xdy$. Then let $P_1$ be the set of elements in $P \setminus P_0$ which are indecomposable with respect to this composition $\circ$. Let $\Delta$ be the set of subsets $\delta = \{s, t, r\} \subseteq S$ such that $m_{st} = m_{tr} = m_{sr} = 3$ and $L(sts) = K_0$.

Let $\tilde{\mathscr{C}}(W)$ be the category with the following data: objects are pairs $(d, b)$ with $d \in P_0$, $b \in B_d$; morphisms are given by $\mathrm{Hom}_{\mathscr{C}}((d, b), (d', b')) = P_{d',d}$; the composition of morphisms is given by $x \circ y$ for $x, y \in P$.

Let $\tilde{\mathscr{D}}(W)$ be the *preadditive* category with the following data: objects are pairs $(d, b)$ with $d \in P_0$, $b \in B_d$; morphisms are given by $\mathrm{Hom}_{\mathscr{D}}((d, b), (d', b')) = \bigoplus_{x \in P_{d',d}} \mathbb{Z} f_x$; the composition of morphisms is given by

$$f_x f_y = \sum_{z \in P} \gamma_{x,y,z^{-1}} f_z$$

where $\gamma_{x,y,z^{-1}}$ $(x, y, z \in P)$ is the leading term coefficient of $h_{x,y,z}$, and $C_x C_y = \sum_{z \in P} h_{x,y,z} C_z$.

By Lemma 3.12, we only need to consider the *full* subcategory $\mathscr{C}(W)$ (resp. $\mathscr{D}(W)$) of $\tilde{\mathscr{C}}(W)$ (resp. $\tilde{\mathscr{D}}(W)$) with objects $(d, e)$, $d \in P_0$, where $e$ is the neutral element of $W$.

For $\delta \in \Delta$, let $P_\delta = P \cap W_\delta$, and let $\mathscr{C}_\delta$ be the subcategory (not full) of $\mathscr{C}(W)$ with morphisms given by $P_\delta$. The corresponding subcategory of $\mathscr{D}(W)$ is denoted by $\mathscr{D}_\delta$. For