Reduced fusion systems over 2-groups of small order

Kasper K.S. Andersen, Bob Oliver, Joana Ventura

PII: S0021-8693(17)30360-5
DOI: http://dx.doi.org/10.1016/j.jalgebra.2017.06.016
Reference: YJABR 16271

To appear in: *Journal of Algebra*

Received date: 16 June 2016

REDUCED FUSION SYSTEMS OVER 2-GROUPS OF SMALL ORDER

KASPER K. S. ANDERSEN, BOB OLIVER, AND JOANA VENTURA

Abstract. We prove, when S is a 2-group of order at most 2^9, that each reduced fusion system over S is the fusion system of a finite simple group and is tame. It then follows that each saturated fusion system over a 2-group of order at most 2^9 is realizable. What is most interesting about this result is the method of proof: we show that among 2-groups with order in this range, the ones which can be Sylow 2-subgroups of finite simple groups are almost completely determined by criteria based on Bender's classification of groups with strongly 2-embedded subgroups.

A *saturated fusion system* over a finite p-group S is a category whose objects are the subgroups of S, whose morphisms are monomorphisms between subgroups, and which satisfy certain axioms first formulated by Puig [Pg] and motivated in part by conjugacy relations among p-subgroups of a given finite group. A saturated fusion system is *realizable* if it is isomorphic to the fusion system defined by the conjugation relations within a Sylow p-subgroup of some finite group, and is *exotic* otherwise. One of our main goals is to try to understand when and how exotic fusion systems can occur, especially over 2-groups.

A saturated fusion system $\mathcal{F}$ is *reduced* if $O_p(\mathcal{F}) = 1$ and $O^p(\mathcal{F}) = O^{p'}(\mathcal{F}) = \mathcal{F}$ (see Definitions 1.1(c,e) and 1.9(a)). A saturated fusion system $\mathcal{F}$ is *tame* if it is realized by a group G such that the natural homomorphism from $\mathrm{Out}(G)$ to a certain group of outer automorphisms of $\mathcal{F}$ (more precisely, of an associated linking system) is split surjective (Definition 1.10). The main result in our earlier paper [AOV1] says roughly that exotic fusion systems can be detected via tameness of associated reduced fusion systems. More precisely, by [AOV1, Theorems A & B], if the "reduction" of a fusion system $\mathcal{F}$ is tame, then $\mathcal{F}$ is tame and hence realizable, while if a reduced fusion system is not tame, then it is the reduction of an exotic fusion system.

A saturated fusion system is *indecomposable* if it does not split as a product of fusion systems over nontrivial p-groups. We can now state our main result.

Theorem A. *Let $\mathcal{F}$ be a reduced, indecomposable fusion system over a nontrivial 2-group of order at most 2^9. Then $\mathcal{F}$ is the fusion system of a finite simple group, and is tame.*

Proof. This is shown in Theorems 4.1 (for 2-groups of order at most 64), 4.3 (order 2^7), 5.1 (order 2^8), and 6.1 (order 2^9). □

The next theorem follows from Theorem A and the above discussion.

Theorem B. *Each saturated fusion system over a 2-group of order at most 2^9 is realizable.*

2000 *Mathematics Subject Classification.* Primary 20D20. Secondary 20D05, 20E45, 20–04.

Key words and phrases. Finite simple groups, fusion, Sylow subgroups.

K. K. S. Andersen was partially supported by the Danish National Research Foundation (DNRF) through the Centre for Symmetry and Deformation, and by VILLUM FONDEN through the network for Experimental Mathematics in Number Theory, Operator Algebras, and Topology.

B. Oliver is partially supported by UMR 7539 of the CNRS, and by project ANR BLAN08-2_338236, HGRT.

J. Ventura was partially supported by FCT through program POCI 2010/FEDER and project PTDC/MAT/098317/2008.