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On the Stanley depth of powers of edge ideals

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ABSTRACT

Let $\mathbb{K}$ be a field and $S = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring in n variables over $\mathbb{K}$. Let G be a graph with n vertices. Assume that $I = I(G)$ is the edge ideal of G and p is the number of its bipartite connected components. We prove that for every positive integer k, the inequalities $\mathrm{sdepth}(I^k/I^{k+1}) \geq p$ and $\mathrm{sdepth}(S/I^k) \geq p$ hold. As a consequence, we conclude that S/I^k satisfies Stanley's inequality for every integer $k \geq n-1$. Also, it follows that I^k/I^{k+1} satisfies Stanley's inequality for every integer $k \gg 0$. Furthermore, we prove that if (i) G is a non-bipartite graph, or (ii) at least one of the connected components of G is a tree with at least one edge, then I^k satisfies Stanley's inequality for every integer $k \geq n-1$. Moreover, we verify a conjecture of the author in special cases.

1. Introduction

Let $\mathbb{K}$ be a field and let $S = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring in n variables over $\mathbb{K}$. Let M be a finitely generated $\mathbb{Z}^n$-graded S-module. Let $u \in M$ be a homogeneous element and $Z \subseteq \{x_1, \dots, x_n\}$. The $\mathbb{K}$-subspace $u\mathbb{K}[Z]$ generated by all elements uv, with v a

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monomial in $\mathbb{K}[Z]$, is called a *Stanley space* of dimension $|Z|$, if it is a free $\mathbb{K}[Z]$-module. Here, as usual, $|Z|$ denotes the number of elements of Z. A decomposition $\mathcal{D}$ of M as a finite direct sum of Stanley spaces is called a *Stanley decomposition* of M. The minimum dimension of a Stanley space in $\mathcal{D}$ is called the *Stanley depth* of $\mathcal{D}$ and is denoted by $\mathrm{sdepth}(\mathcal{D})$. The quantity

$$\mathrm{sdepth}(M) := \max\big\{\mathrm{sdepth}(\mathcal{D}) \mid \mathcal{D} \text{ is a Stanley decomposition of } M\big\}$$

is called the *Stanley depth* of M. We say that a $\mathbb{Z}^n$-graded S-module M satisfies *Stanley's inequality* if

$$\mathrm{depth}(M) \leq \mathrm{sdepth}(M).$$

In fact, Stanley [16] conjectured that every $\mathbb{Z}^n$-graded S-module satisfies Stanley's inequality. This conjecture has been recently disproved in [1]. However, it is still interesting to find the classes of $\mathbb{Z}^n$-graded S-modules which satisfy Stanley's inequality. For a reader friendly introduction to Stanley depth, we refer to [10] and for a nice survey on this topic, we refer to [6].

Let G be a graph with vertex set $V(G) = \big\{v_1, \ldots, v_n\big\}$. The edge ideal $I(G)$ of G is the ideal of S generated by the squarefree monomials x_ix_j, where $\{v_i, v_j\}$ is an edge of G. In [12], the authors proved that if G is a forest (i.e., a graph with no cycle), then $S/I(G)^k$ satisfies Stanley's inequality for every integer $k \gg 0$. Also, it was shown in [2] that $I(G)^k/I(G)^{k+1}$ satisfies Stanley's inequality for every forest G and every integer $k \gg 0$. The aim of this paper is to extend theses results to the whole class of graphs. In Theorem 2.3, we prove that for every graph G, the inequality $\mathrm{sdepth}(S/I(G)^k) \geq p$ holds, where p is the number of bipartite components of G. Combining this inequality with a recent result of Trung [17], we conclude that $S/I(G)^k$ satisfies Stanley's inequality for every integer $k \geq n - 1$ (see Corollary 2.5). In Theorem 2.2, we study the Stanley depth of $I(G)^k/I(G)^{k+1}$ and prove that $\mathrm{sdepth}(I(G)^k/I(G)^{k+1}) \geq p$, for every integer $k \geq 0$. Combining this inequality with a result of Herzog and Hibi [7], we deduce that $I(G)^k/I(G)^{k+1}$ satisfies Stanley's inequality for large k (see Corollary 2.6).

In Section 2, we investigate the Stanley depth of $I(G)^k$, for a positive integer k. In Theorem 3.1, we determine a lower bound for the Stanley depth of $I(G)^k$. In Corollaries 3.2 and 3.5, we prove that if (i) G is a non-bipartite graph, or (ii) at least one of the connected components of G is a tree (i.e., a connected forest) with at least one edge, then for every positive integer k, the Stanley depth of $I(G)^k$ is at least one more than the number of bipartite connected components of G. Then we conclude that for theses classes of graphs, the ideal $I(G)^k$ satisfies Stanley's inequality, for every $k \geq n-1$, where $n =\mid V(G) \mid$ (see Corollary 3.6).