Invariants of Cohen-Macaulay rings associated to their canonical ideals

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INVARIANTS OF COHEN-MACAULAY RINGS ASSOCIATED TO THEIR CANONICAL IDEALS

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Abstract. The purpose of this paper is to introduce new invariants of Cohen-Macaulay local rings. Our focus is the class of Cohen-Macaulay local rings that admit a canonical ideal. Attached to each such ring $\mathbf{R}$ with a canonical ideal $\mathcal{C}$, there are integers–the type of $\mathbf{R}$, the reduction number of $\mathcal{C}$–that provide valuable metrics to express the deviation of $\mathbf{R}$ from being a Gorenstein ring. We enlarge this list with other integers–the roots of $\mathbf{R}$ and several canonical degrees. The latter are multiplicity based functions of the Rees algebra of $\mathcal{C}$.

Key Words and Phrases: Canonical degree, Cohen-Macaulay type, analytic spread, roots, reduction number.

1. Introduction

Let $\mathbf{R}$ be a Cohen-Macaulay ring of dimension $d \geq 1$. If $\mathbf{R}$ admits a canonical module $\mathcal{C}$ and has a Gorenstein total ring of fractions, we may assume that $\mathcal{C}$ is an ideal of $\mathbf{R}$. In this case, we introduce new numerical invariants for $\mathbf{R}$ that refine and extend for local rings the use of its Cohen-Macaulay *type*, the minimal number of generators of $\mathcal{C}$, $r(\mathbf{R}) = \nu(\mathcal{C})$. Among numerical invariants of the isomorphism class of $\mathcal{C}$ are the *analytic spread* $\ell(\mathcal{C})$ of $\mathcal{C}$ and attached reduction numbers. We also introduce the *rootset* of $\mathbf{R}$, which may be a novel invariant. Certain constructions on $\mathcal{C}$, such as the Rees algebra $\mathbf{R}[\mathcal{C}\mathbf{T}]$ leads to an invariant of $\mathbf{R}$, but the associated graded ring $\mathrm{gr}_{\mathcal{C}}(\mathbf{R})$ does not. It carries however properties of a semi-invariant which we will make use of to build true invariants. Combinations of semi-invariants are then used to build invariants under the general designation of *canonical degrees*. The main effort is setting the foundations of the new invariants and examining their relationships. We shall also experiment in extending the construction to more general rings. When we do so, to facilitate the discussion we assume that $\mathbf{R}$ is a homomorphic image of a Gorenstein ring. In a sequel we make applications to Rees algebras, monomial rings, Stanley-Reisner rings.

In Section 2 we introduce our basic canonical degree and derive some of its most direct properties. It requires knowledge of the Hilbert coefficients $\mathrm{e}_0(\cdot)$ of $\mathfrak{m}$-primary ideals:

Theorem 2.2. Let $(\mathbf{R}, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 1$ that has a canonical ideal $\mathcal{C}$. Then

$$\mathrm{cdeg}(\mathbf{R}) = \sum_{\text{height } \mathfrak{p}=1} \mathrm{cdeg}(\mathbf{R}_{\mathfrak{p}}) \deg(\mathbf{R}/\mathfrak{p}) = \sum_{\text{height } \mathfrak{p}=1} [\mathrm{e}_0(\mathcal{C}_{\mathfrak{p}}) - \lambda((\mathbf{R}/\mathcal{C})_{\mathfrak{p}})] \deg(\mathbf{R}/\mathfrak{p})$$

is a well-defined finite sum independent of the chosen canonical ideal $\mathcal{C}$. In particular, if $\mathcal{C}$ is equimultiple with a minimal reduction (a), then

$$\mathrm{cdeg}(\mathbf{R}) = \deg(\mathcal{C}/(a)) = \mathrm{e}_0(\mathfrak{m}, \mathcal{C}/(a)).$$

Two of its consequences when $\mathcal{C}$ is equimultiple are: (i) $\mathrm{cdeg}(\mathbf{R}) \geq r(\mathbf{R}) - 1$; (ii) $\mathrm{cdeg}(\mathbf{R}) = 0$ if and only if $\mathbf{R}$ is Gorenstein.

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