Journal of Algebra 483 (2017) 58–70

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Some remarks about Mishchenko–Fomenko subalgebras

Alexey Bolsinov [a,b]

[a] *Department of Mathematical Sciences, Loughborough University, Loughborough LE11 3TU, UK*
[b] *Faculty of Mechanics and Mathematics, Moscow State University, Moscow 119991, Russia*

ARTICLE INFO

Article history:
Received 1 February 2017
Available online 11 April 2017
Communicated by Michel Van den Bergh

Keywords:
Lie algebra
Coadjoint representation
Poisson bracket
Mishchenko–Fomenko algebras

ABSTRACT

We discuss and compare two different approaches to the notion of Mishchenko–Fomenko subalgebras in Poisson–Lie algebras of finite-dimensional Lie algebras. One of them, commonly accepted by the algebraic community, uses polynomial Ad^*-invariants. The other is based on formal Ad^*-invariants and allows one to deal with arbitrary Lie algebras, not necessarily algebraic. In this sense, the latter is more universal.

1. Introduction

This note is primarily motivated by the paper by A. Ooms [10] in which, among other interesting results, the author constructs a counterexample to my completeness criterion for Mishchenko–Fomenko subalgebras [2]. I do not intend to disprove this statement by Ooms. My point is that the example by A. Ooms and the completeness criterion from [2] are both correct. The confusion is caused by the fact that the definitions of Mishchenko–Fomenko subalgebras used in [10] and [2] are different. The purpose of the

E-mail address: a.bolsinov@lboro.ac.uk.

http://dx.doi.org/10.1016/j.jalgebra.2017.03.032
0021-8693/

present note is to clarify this issue and perhaps to convince the reader that the definition from [2] is, in some respect, better.

2. Formal Ad*-invariants

Originally, Mishchenko–Fomenko subalgebras appeared in the context of integrable Hamiltonian systems on Lie algebras, real or complex. However this construction is purely algebraic and in what follows we consider finite-dimensional Lie algebras over an algebraically closed field $\mathbb{K}$ of characteristic zero. Here we recall and slightly modify the results of [6]. Basically, we want to develop some algebraic techniques allowing us to deal with arbitrary Lie algebras $\mathfrak{g}$, not necessarily algebraic. So we do not assume the existence of any polynomial and even rational Ad^*-invariants. Moreover, we never use the Lie group G associated with $\mathfrak{g}$.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra, $\mathfrak{g}^*$ its dual space and $P(\mathfrak{g})$ denote the algebra of polynomials[1] on $\mathfrak{g}^*$. The algebra $P(\mathfrak{g})$ is endowed with the standard Lie–Poisson bracket

$$\{f(x), g(x)\} = \langle x, [df(x), dg(x)]\rangle, \quad x \in \mathfrak{g}^*, \ df(x), dg(x) \in \mathfrak{g}, \tag{1}$$

and we will refer to $P(\mathfrak{g})$ as the Lie–Poisson algebra associated with $\mathfrak{g}$.

Our goal is to construct a "big" commutative subalgebra in $P(\mathfrak{g})$. The argument shift method suggested by A. Mishchenko and A. Fomenko [9] is based on some nice properties of Ad^*-invariants. In general, however, polynomial (and even rational) invariants do not necessarily exist. To avoid this problem one can use formal invariants which can be defined in the following way.

Definition 1. Let $\mathsf{F} = \sum_{k=1}^{\infty} f^{(k)}$ be a formal power series where $f^{(k)} \in P(\mathfrak{g})$ is a homogeneous polynomial of degree $k \in \mathbb{N}$. We say that F is a *formal* Ad^**-invariant* at a point $a \in \mathfrak{g}^*$, if the following (formal) identity holds for all $\xi \in \mathfrak{g}$:

$$\langle d\mathsf{F}(x), \mathrm{ad}^*_\xi(a + x)\rangle = 0. \tag{2}$$

From the differential-geometric point of view this condition simply means that the differential of F at the point $a + x$ vanishes on the tangent space of the coadjoint orbit through this point. Thus, the above relation can be understood as the standard definition of an invariant function F where F is replaced by its Taylor expansion F at the point $a \in \mathfrak{g}^*$. The formal identity (2) amounts to the following infinite sequence of polynomial relations:

[1] $P(\mathfrak{g})$, as a set, is of course the same as the symmetric Lie algebra $S(\mathfrak{g})$, but we use a slightly different point of view thinking of $P(\mathfrak{g})$ as a Poisson algebra and of its elements as functions on the vector space $\mathfrak{g}^*$.