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Locally nilpotent skew derivations with central invariants

Jeffrey Bergen [a,*], Piotr Grzeszczuk [b,1]

[a] *Department of Mathematics, DePaul University, 2320 N. Kenmore Avenue, Chicago, IL 60614, USA*
[b] *Faculty of Computer Science, Białystok University of Technology, Wiejska 45A, 15-351 Białystok, Poland*

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ABSTRACT

Let δ be a locally nilpotent q-skew derivation of an algebra R such that the invariants are central. With some natural assumptions on the q-characteristic, we show that if R is semiprime then R is commutative. We also examine other conditions which imply, even when R is not commutative, that the commutator ideal is contained in the prime radical. These results extend previous work of the authors and of Osterburg and may shed some light on a conjecture of Herstein.

* Corresponding author.
E-mail addresses: jbergen@depaul.edu (J. Bergen), p.grzeszczuk@pb.edu.pl (P. Grzeszczuk).

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1. Introduction

In [1] it is shown that if δ is an algebraic skew derivation of a semiprime algebra R, such that the invariants are central, then R must be commutative. This generalized a result in [7] on automorphisms of prime order.

In this paper we turn our attention to locally nilpotent q-skew derivations with central invariants. When contrasting the structure of an algebra to the invariants of a transformation such as an automorphism, derivation, or skew derivation, one typically assumes that the transformation is algebraic. One of the surprising aspects of this paper is that we only need assume that our q-skew derivations are locally nilpotent. Along these lines, the main result of this paper is

Theorem 3. *Let δ be a locally nilpotent q-skew derivation of a semiprime algebra R such that the invariants of δ are central. If $\delta^N = 0$, assume $char_q(R) \geq N$ or $char_q(R) = 0$, whereas if δ is not nilpotent, assume $char_q(R) = 0$. Then R is commutative.*

In Theorems 4 and 6, we also consider the situation where R need not be semiprime. Starting in the 1970's, a great deal of machinery, such as the Bergman–Isaacs Theorem [4] and Kharchenko's work [6] on inner and outer actions, was developed and used to contrast the structure of algebras to the invariants under the actions of groups and Lie algebras. A second surprising aspect of this paper is that our arguments are self-contained and do not require any previous results on invariants of automorphisms, derivations, or skew derivations.

In [2] a strong connection is shown between skew polynomial rings and rings with locally nilpotent skew derivations. Motivated by the ideas in that paper, we show that, in many cases, if an algebra R has a locally nilpotent q-skew derivation with central invariants, then there is a large subset C of the center such that if $a, b \in R$, then there exists $c \in C$ such that $c[a, b] = 0$. Therefore, the proofs in this paper deal primarily with proving the existence of such central elements c and then examining their annihilators.

We will now introduce the terminology that will be used throughout this paper. R will be an algebra over a field K and q will be a nonzero element of K. If σ is a K-linear automorphism of R, we say that a K-linear map δ is a q-skew derivation if

$$\delta(rs) = \delta(r)s + \sigma(r)\delta(s) \quad \text{and} \quad \delta(\sigma(r)) = q\sigma(\delta(r)),$$

for all $r, s \in R$. Observe that if $\sigma = 1$, then δ is an ordinary derivation, whereas if $\delta = \sigma - 1$, then δ is q-skew with $q = 1$.

We say that δ is locally nilpotent, if for each $r \in R$, there exists $n = n(r) \geq 1$ such that $\delta^n(r) = 0$. It then follows that if we let $R_n = \{r \in R \mid \delta^{n+1}(r) = 0\}$, then

$$R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots,$$

$R = \bigcup_{n \geq 0} R_n$, R_0 is the invariants of δ, and R_1 is the kernel of δ^2.