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FREE SUBALGBRAS OF GRADED ALGEBRAS

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Abstract. Let k be a field and let $A = \bigoplus_{n\geq 1} A_n$ be a positively graded k-algebra. We recall that A is graded nilpotent if for every $d \geq 1$, the subalgebra of A generated by elements of degree d is nilpotent. We give a method of producing grading nilpotent algebras and use this to prove that over any base field k there exists a finitely generated graded nilpotent algebra that contains a free k-subalgebra on two generators.

1. Introduction

In recent years there has been a flurry of activity in ring theory around problems related to the Kurosh and Köthe conjectures (see, for example, [8, 14, 18, 19, 23, 24]). The Kurosh conjecture, which asserts that a finitely generated algebra that is algebraic over the base field is necessarily finite-dimensional, was disproved in 1964 by Golod and Shafarevich [11], who constructed a counter-example to both this conjecture and to the Burnside problem, which is the group theoretic analogue of the Kurosh problem. The Köthe conjecture, on the other hand is still open. It asks whether the sum of two nil left ideals in a ring is again a nil left ideal. This was proved for uncountable base fields by Amitsur [2] in 1956, but the conjecture remains open in general.

Köthe's conjecture is equivalent to the statement that if R is a nil ring then $R[x]$ is equal to its own Jacobson radical (see Krempa [13] for this and other equivalent statements). Towards better understanding this conjecture, Amitsur [3] conjectured that if R is nil then in fact $R[x]$ is nil, too, which is a stronger conjecture. Again, Amitsur proved that this is true when R is an algebra over an uncountable base field [4], but Smoktunowicz [22] later constructed counter-examples to Amitsur's conjecture over countable fields. Her constructions have since provided impetus for a lot of the recent work around nil rings and questions inspired by Köthe's conjecture and related problems.

We observe that if one takes a nil ring R and one grades $R[x]$ by letting the elements of R be of degree zero and letting the central variable x have degree one, then while Smoktunowicz has shown that $R[x]$ need not be nil,

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