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SHIFTED SYMMETRIC FUNCTIONS AND MULTIRECTANGULAR COORDINATES OF YOUNG DIAGRAMS

PER ALEXANDERSSON AND VALENTIN FÉRAY

Abstract. In this paper, we study shifted Schur functions $S^\star_\mu$, as well as a new family of shifted symmetric functions $\mathfrak{K}_\mu$ linked to Kostka numbers. We prove that both are polynomials in multi-rectangular coordinates, with nonnegative coefficients when written in terms of falling factorials.

We then propose a conjectural generalization to the Jack setting. This conjecture is a lifting of Knop and Sahi's positivity result for usual Jack polynomials and resembles recent conjectures of Lassalle. We prove our conjecture for one-part partitions.

1. Introduction

We use standard notation for partitions and symmetric functions, which is recalled in Section 2.

1.1. **Shifted symmetric functions.** Informally, a shifted symmetric function is a formal power series in infinitely many variables x_1, x_2, … that has bounded degree and is symmetric in the "shifted" variables $x_1 - 1$, $x_2 - 2$, … (a formal definition is given in Section 2.5).

Many properties of symmetric functions have natural analogue in the shifted framework. Unlike in symmetric function theory, it is often relevant to evaluate a shifted symmetric function F on the parts of a Young diagram $\lambda = (\lambda_1, \dots, \lambda_\ell)$. Then we denote $F(\lambda) := F(\lambda_1, \dots, \lambda_\ell, 0, 0, \dots)$. It turns out that shifted symmetric functions are determined by their image on Young diagrams, so that the shifted symmetric function ring will be identified with a subalgebra of the algebra of functions on the set of all Young diagrams (without size nor length restriction).

Shifted symmetric functions were introduced by Okounkov and Olshanski in [OO97b]. In this paper, the authors are particularly interested in the basis of *shifted Schur functions*, which can be defined as follows: for any integer partition μ and any $n \geq 1$,

$$S^\star_\mu(x_1, \dots, x_n) = \frac{\det\left((x_i + n - i)_{\mu_j + n - j}\right)}{\det\left((x_i + n - i)_{n-j}\right)},$$

where $(x)_k$ denotes the falling factorial $x(x-1)\cdots(x-k+1)$. Note the similarity with the definition of Schur functions [Mac95, p. 40]: in particular, the highest degree terms of $S^\star_\mu$ is the Schur function S_μ. Shifted Schur functions are also closely related to *factorial Schur polynomials*, originally defined by Biedenharn and Louck in [BL89] and further studied, *e.g.*, in [Mac92, MS99]. These functions display beautiful properties:

- Some well-known formulas involving Schur functions have a natural extension to shifted Schur functions, *e.g.*, the combinatorial expansion in terms of semi-standard Young tableaux [OO97b, Theorem 11.1] and the Jacobi-Trudi identity [OO97b, Theorem 13.1].
- The evaluation $S^\star_\mu(\lambda)$ of a shifted Schur function indexed by μ on a Young diagram λ has a combinatorial meaning: it vanishes if λ does not contain μ and is related to the number of standard Young tableaux of skew shape λ/μ otherwise; see [OO97b, Theorem 8.1]. Note that this beautiful property has no analogue for usual (*i.e.*, non-shifted) symmetric functions.
- Lastly, they appear as eigenvalues of elements of well-chosen bases in highest weight modules for classical Lie groups, see [OO98].

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