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Fano 4-folds, flips, and blow-ups of points

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ABSTRACT

We study smooth, complex Fano 4-folds X with large Picard number ρ_X, with techniques from birational geometry. Our main result is that if X is isomorphic in codimension one to the blow-up of a smooth projective 4-fold Y at a point, then $\rho_X \leq 12$. We give examples of such X with Picard number up to 9. The main theme (and tool) is the study of fixed prime divisors in Fano 4-folds, especially in the case $\rho_X > 6$, in which we give some general results of independent interest.

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1. Introduction

In this paper we study (smooth, complex) Fano 4-folds X with large Picard number ρ_X. Let us recall that ρ_X is equal to the second Betti number of X, and since there are finitely many families of Fano 4-folds, ρ_X is bounded. We also have an explicit bound on ρ_X (see [11, Remark 3.1]), which however is, conjecturally, far from being sharp; the maximal value of ρ_X for a Fano 4-fold X is not yet known.

As for examples, taking products of del Pezzo surfaces one gets Fano 4-folds with Picard number up to 18; to the author's knowledge, the other known examples of Fano 4-folds have $\rho \leq 9$. In fact, for $\rho = 7, 8, 9$, the author is aware of only one family (for each ρ) of Fano 4-folds which is not a product of surfaces, obtained as follows. Consider the blow-up $\mathrm{Bl}_{p_1,\dots,p_r}\mathbb{P}^4$ of $\mathbb{P}^4$ in r general points. For $r = 2, \dots, 8$ this variety is not Fano, but can be modified with a finite sequence of flips[1] in order to get a smooth Fano 4-fold X with $\rho_X = 1 + r \leq 9$; we refer the reader to Example 6.1 for more details.

The main object of this paper is the study of Fano 4-folds obtained as in the previous example: by the blow-up of a (smooth) point, followed by a sequence of flips.

Let us first recall that Fano manifolds that can be obtained by blowing-up a point in another manifold have been classified, in arbitrary dimension ≥ 3, by Bonavero, Campana, and Wiśniewski [8]; in particular such X always has $\rho_X \leq 3$.

In the case where we allow also flips, our main result is a bound on the Picard number.

Theorem 1.1. *Let X be a smooth Fano 4-fold. Suppose that there exist a normal and $\mathbb{Q}$-factorial projective variety Y, and a smooth point $p \in Y$, such that X and $\mathrm{Bl}_p Y$ are isomorphic in codimension one. Then $\rho_X \leq 12$.*

[1] These are K-positive flips, in the terminology of this paper.