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The diameter of the generating graph of a finite soluble group

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ABSTRACT

Let G be a finite 2-generated soluble group and suppose that $\langle a_1, b_1\rangle = \langle a_2, b_2\rangle = G$. Then there exist c_1, c_2 such that $\langle a_1, c_1\rangle = \langle c_1, c_2\rangle = \langle c_2, a_2\rangle = G$. Equivalently, the subgraph $\Delta(G)$ of the generating graph of a 2-generated finite soluble group G obtained by removing the isolated vertices has diameter at most 3. We construct a 2-generated group G of order $2^{10} \cdot 3^2$ for which this bound is sharp. However a stronger result holds if G' has odd order or G' is nilpotent: in this case there exists $b \in G$ with $\langle a_1, b\rangle = \langle a_2, b\rangle = G$.

1. Introduction

Let G be a finite group. The generating graph for G, written $\Gamma(G)$, is the graph where the vertices are the nonidentity elements of G and there is an edge between g_1 and g_2 if G is generated by g_1 and g_2. If G is not 2-generated, then there will be no edge in this graph. Thus, it is natural to assume that G is 2-generated.

Quite a lot is known about this graph when G is a nonabelian simple group; for example Guralnick and Kantor [10] showed that there is no isolated vertex in $\Gamma(G)$ and Breuer, Guralnick, Kantor [2] showed that the diameter of $\Gamma(G)$ is 2 for all G.

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On the other hand, relatively little is known in the general case (for arbitrary finite groups). There could be many isolated vertices in this graph. All of the elements in the Frattini subgroup will be isolated vertices, but we can also find isolated vertices outside the Frattini subgroup (for example the nontrivial elements of the Klein subgroup are isolated vertices in $\Gamma(\mathrm{Sym}(4))$).

Let $\Delta(G)$ be the subgraph of $\Gamma(G)$ that is induced by all the vertices that are not isolated. In [5] it is proved that if G is a 2-generated soluble group, then $\Delta(G)$ is connected. In this paper we investigate the diameter $\mathrm{diam}(\Delta(G))$ of this graph.

Theorem 1. *If G is a 2-generated finite soluble group, then $\Delta(G)$ is connected and* $\mathrm{diam}(\Delta(G)) \leq 3$.

The situation is completely different if the solubility assumption is dropped. It is an open problem whether or not $\Delta(G)$ is connected, but even when $\Delta(G)$ is connected, its diameter can be arbitrarily large. For example if G is the largest 2-generated direct power of $\mathrm{SL}(2, 2^p)$ and p is a sufficiently large odd prime, then $\Delta(G)$ is connected but $\mathrm{diam}(\Delta(G)) \geq 2^{p-2} - 1$ (see [3, Theorem 5.4]).

For soluble groups, the bound $\mathrm{diam}(\Delta(G)) \leq 3$ given in Theorem 1 is best possible. In Section 3 we construct a soluble 2-generated group G of order $2^{10} \cdot 3^2$ with $\mathrm{diam}(\Delta(G)) = 3$. However we prove that $\mathrm{diam}(\Delta(G)) \leq 2$ in some relevant cases.

Theorem 2. *Suppose that a finite 2-generated soluble group G has the property that $|\mathrm{End}_G(V)| > 2$ for every nontrivial irreducible G-module V which is G-isomorphic to a complemented chief factor of G. Then $\mathrm{diam}(\Delta(G)) \leq 2$, i.e. if $\langle a_1, b_1\rangle = \langle a_2, b_2\rangle = G$, then there exists $b \in G$ with $\langle a_1, b\rangle = \langle a_2, b\rangle = G$.*

Corollary 3. *Let G be a 2-generated finite group. If the derived subgroup of G has odd order, then* $\mathrm{diam}(\Delta(G)) \leq 2$.

Corollary 4. *Let G be a 2-generated finite group. If the derived subgroup of G is nilpotent, then* $\mathrm{diam}(\Delta(G)) \leq 2$.

2. Proof of Theorem 2

We prove Theorem 2 with the same approach that will be used in the proof of Theorem 1. However we prefer to give in this particular case an easier and shorter argument. Before doing that, we briefly recall some necessary definitions and results. Given a subset X of a finite group G, we will denote by $d_X(G)$ the smallest cardinality of a set of elements of G generating G together with the elements of X. The following generalizes a result originally obtained by W. Gaschütz [8] for $X = \varnothing$.