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On the number of near-vector spaces determined by finite fields

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ABSTRACT

A mistake on a paper concerning near-vector spaces is fixed. A new characterization of near-vector spaces determined by finite fields is provided and the number (up to isomorphism) of these spaces is exhibited.

1. Introduction

In 1974, Andre introduced and studied the concept of near-vector spaces. Later several researchers, for example, Van der walt, Howell, Meyer and Tim Boykett, devoted time into investigating such concepts. In 2010, Howell and Meyer classified near-vector spaces over finite fields of p (p is prime) elements up to isomorphism. They also extended the result to a finite field of p^n elements in Theorem 3.9, [2]. This theorem asserts that the number of near-vector spaces $V = \mathbb{F}^{\oplus m}$ over a finite field $\mathbb{F} = GF(p^n)$ is exactly

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$$\binom{m + \frac{\phi(p^n-1)}{n} - 2}{m-1}$$

up to isomorphism in Definition 2.2, where ϕ is the Euler's totient function. This number is calculated based on the number of distinct suitable sequences (Definition 2.3 in [2]). Namely, if A_1^* and A_2^* are determined by suitable sequences (S_1) and (S_2) with 1 in the first position, respectively, then $(\mathbb{F}^{\oplus m}, A_1^*) \cong (\mathbb{F}^{\oplus m}, A_2^*)$ if and only if $(S_1) = (S_2)$.

However, for the case $m = 4$, $n = 3$ and $p = 3$, it turns out that $(\mathbb{F}^4, A_1^*) \cong (\mathbb{F}^4, A_2^*)$, where $A_1^* = \{s_\alpha \,|\, \alpha \in \mathbb{F}\}$ and $A_2^* = \{t_\beta \,|\, \beta \in \mathbb{F}\}$ are constructed using the sequences $(S_1) = (1, 1, 5, 5)$ and $(S_2) = (1, 1, 7, 7)$, respectively, which are distinct suitable sequences. Precisely, the isomorphism is obtained by the group isomorphisms $\theta : \mathbb{F}^{\oplus 4} \longrightarrow \mathbb{F}^{\oplus 4}$ defined by $\theta(x_1, x_2, x_3, x_4) := (x_3, x_4, x_1^9, x_2^9)$ and $\eta : A_1^* \longrightarrow A_2^*$ defined by $\eta(s_\alpha) := t_{\alpha^5}$, and it can be shown that

$$\begin{aligned}
\theta((x_1, x_2, x_3, x_4)s_\alpha) &= \theta(x_1\alpha, x_2\alpha, x_3\alpha^5, x_4\alpha^5) \\
&= (x_3\alpha^5, x_4\alpha^5, x_1^9\alpha^9, x_2^9\alpha^9) \\
&= (x_3\alpha^5, x_4\alpha^5, x_1^9(\alpha^5)^7, x_2^9(\alpha^5)^7) \\
&= (x_3, x_4, x_1^9, x_2^9)t_{\alpha^5} \\
&= \theta(x_1, x_2, x_3, x_4)\eta(s_\alpha)
\end{aligned}$$

for all $(x_1, x_2, x_3, x_4) \in \mathbb{F}^{\oplus 4}, s_\alpha \in A_1^*$. This contradicts the main results of the paper. A slip can be found in the proof of Theorem 3.9 in [2] (line 17 in the proof) in which they are using the isomorphism η as $\eta(s_\alpha) = t_\alpha$. In fact, this should be $\eta(s_\alpha) = t_{\alpha^q}$ for some $1 \le q \le p^n - 1$ and $\gcd(q, p^n - 1) = 1$.

In this article, the slip is fixed and a criteria for the classification of near-vector spaces $\mathbb{F}^{\oplus m}$ over a finite field $\mathbb{F} = GF(p^n)$ is provided. The number of near-vector spaces up to the isomorphism is also displayed based on the subgroup lattice of the abelian group $G := U(p^n - 1)/\langle p \rangle$.

2. Preliminary

Let p be a prime, n a positive integer and $\mathbb{F} = \mathrm{GF}(p^n)$ a field of p^n elements. We first recall the definition of a near-vector space over a finite field $\mathbb{F}$.

Definition 2.1. *([1], cf. [2]) A pair (V, A) is called a* ***near-vector space*** *if:*

(1) $(V, +)$ *is a group and* A *is a set of endomorphisms of* V*;*
(2) A *contains the endomorphisms* 0, *id and* $-id$*;*
(3) $A^* = A \backslash \{0\}$ *is a subgroup of the group* $\mathrm{Aut}(V)$*;*
(4) A *acts fixed point freely on* V, *i.e., for* $x \in V$ *and* $\alpha, \beta \in A$, $x\alpha = x\beta$ *implies that* $x = 0$ *or* $\alpha = \beta$*;*