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Epimorphisms in varieties of residuated structures

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A B S T R A C T

It is proved that epimorphisms are surjective in a range of varieties of residuated structures, including all varieties of Heyting or Brouwerian algebras of finite depth, and all varieties consisting of Gödel algebras, relative Stone algebras, Sugihara monoids or positive Sugihara monoids. This establishes the infinite deductive Beth definability property for a corresponding range of substructural logics. On the other hand, it is shown that epimorphisms need not be surjective in a locally finite variety of Heyting or Brouwerian algebras of width 2. It follows that the infinite Beth property is strictly stronger than the so-called finite Beth property, confirming a conjecture of Blok and Hoogland.

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R-mingle
Beth definability

1. Introduction

A morphism h in a category C is called a (C-) *epimorphism* provided that, for any two C-morphisms f, g from the co-domain of h to a single object,

$$\text{if } f \circ h = g \circ h, \text{ then } f = g.$$

We shall not distinguish notationally between a class K of similar algebras and the concrete category of algebraic homomorphisms between its members. Clearly, in such a category, every surjective K-morphism is a K-epimorphism. If the converse holds, then K is said to have the *epimorphism surjectivity property*, or briefly, the *ES property*.

This property fails, for instance, in the variety of rings. There, the inclusion $\mathbb{Z} \longrightarrow \mathbb{Q}$ is an epimorphism, mainly because multiplicative inverses are 'implicitly defined', i.e., uniquely determined *or* non-existent. The failure of surjectivity reflects the absence of an explicit unary term defining inversehood in the language of rings. In a slogan: epimorphisms correspond to implicit definitions and surjective homomorphisms to explicit ones.

Groups, modules over a given ring, semilattices and lattices each form a variety in which all epimorphisms are surjective; see the references in [34]. The ES property need not persist in subvarieties, however. Indeed, it fails for *distributive* lattices, where an embedding of the three-element chain in a four-element Boolean lattice is an epimorphism (owing to the uniqueness of existent complements).

As this suggests, it is generally difficult to determine whether epimorphisms are surjective in a given variety. Here, for a range of varieties of residuated structures, we shall prove that they are. The ES property is algebraically natural, but our main motivation comes from logic, as residuated structures algebraize substructural logics [21].

The algebraic counterpart K of an algebraizable logic $\vdash$ is a *prevariety*, i.e., a class of similar algebras, closed under isomorphisms, subalgebras and direct products; see [6,7,9,10,18]. In this situation,

> K has the ES property iff $\vdash$ has the *infinite (deductive) Beth (definability) property* [6, Thm. 3.17].

The latter signifies that, in $\vdash$, whenever a set Z of variables is defined *implicitly* in terms of a disjoint set X of variables by means of some set Γ of formulas over $X \cup Z$, then Γ also defines Z *explicitly* in terms of X. In substructural logics, this means, more precisely, that whenever