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Modified double Poisson brackets

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ABSTRACT

We propose a non-skew-symmetric generalization of the original definition of double Poisson Bracket by M. Van den Bergh. It allows one to explicitly construct a more general class of H_0-Poisson structures on finitely generated associative algebras. We show that modified double Poisson brackets inherit certain major properties of the double Poisson brackets.

1. Introduction

Application of the noncommutative geometry program to symplectic manifolds was originated in [9]. Following general philosophy formulated by M. Kontsevich any algebraic property that makes geometric sense is mapped to its commutative counterpart by the functor Rep_N:

$$\mathrm{Rep}_N : \quad \text{fin. gen. Associative algebras} \rightarrow \text{Affine schemes},$$

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which assigns to a finitely generated associative algebra $\mathcal{A}$ a scheme of its' $N \times N$ matrix representations[1]

$$\mathrm{Rep}_N(\mathcal{A}) = Hom(\mathcal{A}, Mat_N(\mathbb{C})).$$

In line with this general philosophy, M. Van den Bergh [7] proposed a definition of the double Poisson bracket on an associative algebra which induces a conventional Poisson bracket on the coordinate ring of matrix representations.

On the contrary, W. Crawley-Boevey [5,6] suggested yet another related definition of the noncommutative analogue of the Poisson bracket, the so-called H_0-Poisson structure. The latter has weaker requirements and in general provides a conventional Poisson bracket only on the moduli space of representations. A double Poisson bracket induces an H_0-Poisson structure but not vice versa.

One of the major advantages of the double Poisson bracket as opposed to an H_0-Poisson structure is that for a finitely generated associative algebra it is defined completely by its action on generators. This allows one to provide numerous explicit examples of double Poisson brackets [16,3] and even carry out certain partial classification problems [13].

In this note we provide an extension of the original ideas of M. Van den Bergh and W. Crawley-Boevey. We define a notion of a *modified* double Poisson bracket (see Definition 1) which allows one to construct explicitly more general examples of H_0-Poisson brackets on finitely generated algebras. We support our definition with new examples of modified double Poisson brackets in Sec. 4 and calculate the corresponding dimensions of symplectic leaves of the induced Poisson structures on the moduli space.

In Section 5 we use the algebra of noncommutative poly-vector fields introduced in [7] to construct non-skew-symmetric biderivations. We introduce the notion of a modified double Poisson bivector and present an essential example.

Finally, in Section 6.1 we investigate brackets on representation algebras induced by the modified double Poisson brackets. We show that some recent results of G. Massuyeau and V. Turaev [11] can be extended beyond skew-symmetric case as well.

2. Modified double Poisson bracket

Let $\mathcal{A} = \mathbb{C}\langle x^{(1)}, \dots, x^{(k)}\rangle/\mathcal{R}$ be an associative algebra over $\mathbb{C}$, which is finitely generated by $\{x^{(1)}, \dots x^{(k)}\}$, possibly with some finite number of relations $\mathcal{R}$.

Definition 1. A modified double Poisson bracket on $\mathcal{A}$ is a map $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ s.t. for all $a, b, c \in \mathcal{A}$

$$\{\!\{a \otimes bc\}\!\} = (b \otimes 1)\{\!\{a \otimes c\}\!\} + \{\!\{a \otimes b\}\!\}(1 \otimes c) \tag{1a}$$

[1] Throughout the text we assume the ground field to be $\mathbb{C}$. All unadorned tensor products, algebras, and schemes are over $\mathbb{C}$ unless specified otherwise.