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Rational maps between varieties associated to central simple algebras

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In this paper we show that there exist rational maps between varieties associated to isomorphism classes of central simple algebras such as Severi–Brauer varieties, norm hypersurfaces and symmetric powers, provided the Brauer-equivalence classes of the algebras involved generate the same cyclic subgroup in $\mathrm{Br}(k)$. In some cases we even get rational embeddings. We relate the obtained results to the Amitsur conjecture.

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1. Introduction

It is well-known that isomorphism classes of central simple algebras are in one-to-one correspondence with isomorphism classes of Severi–Brauer varieties over a field k (see [2]). From this point of view it is natural to study the geometry of a Severi–Brauer variety in terms of the algebraic structure of the corresponding algebra and vice versa. Recall the following theorem of Amitsur [1] (alternatively see [7], Theorem 5.4.1):

Theorem. *Let X be the Severi–Brauer variety corresponding to a central simple k-algebra A. Denote by $F(X)$ the function field of X. Then the kernel of the restriction map $\mathrm{Br}(k) \to \mathrm{Br}(F(X))$, $B \mapsto B \otimes_k F(X)$, is a cyclic group generated by the class $[A]$ of A.*

A direct consequence of this theorem is that the Brauer-equivalence classes of two central simple algebras of same degree generate the same cyclic subgroup in $\mathrm{Br}(k)$ if the corresponding Severi–Brauer varieties are birational. In [1] Amitsur proved that the other implication holds as well, at least for certain fields k. He conjectured that two Severi–Brauer varieties X and Y are birational if and only if the Brauer-equivalence classes of the corresponding central simple algebras A and B generate the same cyclic subgroup in $\mathrm{Br}(k)$. This conjecture is nowadays called the *Amitsur conjecture* for Severi–Brauer varieties. Several results in favor of this conjecture are known (see [1], [13], [15], [20], [24]) and it is also natural to extend the conjecture to products [8]. In view of this conjecture it is an important task to understand how the existence of rational maps between Severi–Brauer varieties is detected by the Brauer group and the algebraic structure of the corresponding central simple algebras.

To give the main results of the present paper, let D and D' be central division algebras of same degree and X and Y the corresponding Severi–Brauer varieties. Denote by $\mathcal{X}$ and $\mathcal{Y}$ the Severi–Brauer varieties corresponding to $M_n(D)$ and $M_n(D')$ for some integer $n > 1$. It is well-known that there are closed immersions $X \hookrightarrow \mathcal{X}$ and $Y \hookrightarrow \mathcal{Y}$, and that if $[D]$ and $[D']$ generate the same cyclic subgroup in $\mathrm{Br}(k)$, one gets dominant rational maps $X \dashrightarrow Y$ and $Y \dashrightarrow X$ (see [12]). But in general, the closed immersions $X \hookrightarrow \mathcal{X}$ and $Y \hookrightarrow \mathcal{Y}$ do not induce rational embeddings, for instance of X into $\mathcal{Y}$. Moreover, it is not clear whether the rational map $X \dashrightarrow Y$ is birational. In this context, we prove the following result:

Theorem *(Theorem 4.3). Let $\mathcal{X}$ and $\mathcal{Y}$ be the Severi–Brauer varieties corresponding to central simple k-algebras $A = M_n(D)$ and $B = M_n(D')$ with $\deg(D) = \deg(D')$ and $n > 1$ arbitrary. Denote by X and Y the Severi–Brauer varieties corresponding to D and D'. Then $\langle [A] \rangle = \langle [B] \rangle$ in $\mathrm{Br}(k)$ if and only if there are rational embeddings $X \dashrightarrow \mathcal{Y}$ and $Y \dashrightarrow \mathcal{X}$.*

By definition, the rational embedding $X \dashrightarrow \mathcal{Y}$ from above can be factored as $U \to Z \to \mathcal{Y}$, where the first arrow is an open and the latter one a closed immersion. Here U