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Which semifields are exact?

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ABSTRACT

Every (left) linear function on a subspace of a finite-dimensional vector space over a (skew) field can be extended to a (left) linear function on the whole space. This paper explores the extent to what this basic fact of linear algebra is applicable to more general structures. Semifields with a similar property imposed on linear functions are called (left) exact, and we present a complete description of such semifields. Namely, we show that a semifield S is left exact if and only if S is either a skew field or an idempotent semiring.

1. Introduction

A set S equipped with two binary operations $+$ and $\cdot$ is called a *semiring* if the following conditions are satisfied: (i) $(R, +)$ is a commutative monoid, (ii) $(R, \cdot)$ is a monoid, (iii) multiplication distributes over addition from both sides, and (iv) the additive identity 0 satisfies $0x = x0 = 0$, for any $x \in R$. In other words, semirings differ from rings by the fact that their elements are not required to have additive inverses. We denote the multiplicative identity by 1, and we assume that $0 \neq 1$. The set S^n becomes a *free left semimodule* if we define the operations $(s_1, \ldots, s_n) \to (\lambda s_1, \ldots, \lambda s_n)$ for all $\lambda \in S$.

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A considerable amount of recent work [9,11,12] is devoted to the concept of so-called *exactness*, which gives a characterization of semirings that behave nicely with respect to basic linear algebraic properties. Namely, a semiring S is called *left exact* if, for every finitely generated left semimodule $L \subseteq S^n$ and every left S-linear function $\varphi : L \to S$, there is a left S-linear function $\varphi_0 : S^n \to S$ that coincides with φ on L. This property becomes a standard result of linear algebra if S is a division ring, so we can conclude that division rings are left exact. The concept of right exactness can be defined dually, and the semirings that are both left and right exact are called simply *exact*. Therefore, the division rings are the first examples of exact semirings. Let us also point out that, in the case of rings, the exactness is equivalent to the property known as *FP-injectivity*, see [3,5,11].

In this paper, we continue studying the semirings in which all the non-zero elements have multiplicative inverses. Such objects form an important class of semirings and are known as *semifields*. Various examples of semifields arise in different applications, and they include the division rings, the semiring of nonnegative reals [13], the tropical semiring [8], the binary Boolean algebra [7], and many others. The aim of our paper is to give a complete characterization of those semifields that are exact.

Theorem 1.1. *Let S be a semifield. Then S is left exact if and only if*

(1) *S is a division ring, or*
(2) *we have $1 + 1 = 1$ in S.*

By symmetry, the conclusion of the theorem holds for right exactness as well. In particular, we get that a semifield is left exact if and only if it is right exact. As a corollary of Theorem 1.1, we get the previously known fact that the tropical semiring $\mathbb{T} = (\mathbb{R} \cup \{\infty\}, \min, +)$ is exact. As far as I can see, the exactness of $\mathbb{T}$ follows from Theorem 5.3 of [6], and I would like to thank the reviewer for pointing my attention to that paper. Corollary 40 in [2] contains a generalization of this result to the class of complete idempotent reflexive semirings. The subsequent paper [12] contains the exactness proofs for other related semirings, including $\overline{\mathbb{T}} = (\mathbb{R} \cup \{+\infty, -\infty\}, \min, +)$. Another proof that $\mathbb{T}$ is exact is contained in the paper [5], where the authors do also prove that an additively cancellative semifield is exact if and only if it is a field.

Our paper is structured as follows. In Section 2, we obtain a useful characterization of exactness resembling some of the results in [12]. We use this characterization (Theorem 2.1) to prove the 'only if' part of Theorem 1.1. In Section 3, we get an improved version of Theorem 2.1 which is valid for semifields. In Section 4, we employ the developed technique and complete the proof of Theorem 1.1. In Section 5, we discuss the perspectives of further work and point out several intriguing open questions.

2. Another characterization of exactness

Let us begin with some notational conventions. We will denote matrices and vectors over a semiring S by bold letters. We denote by $\mathbf{A}_i$ and $\mathbf{A}^j$ the ith row and jth column