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Depth and Stanley depth of symbolic powers of cover ideals of graphs

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ABSTRACT

Let G be a graph with n vertices and let $S = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field $\mathbb{K}$. Assume that $J(G)$ is the cover ideal of G and $J(G)^{(k)}$ is its k-th symbolic power. We prove that the sequences $\{\mathrm{sdepth}(S/J(G)^{(k)})\}_{k=1}^{\infty}$ and $\{\mathrm{sdepth}(J(G)^{(k)})\}_{k=1}^{\infty}$ are non-increasing and hence convergent. Suppose that $\nu_o(G)$ denotes the ordered matching number of G. We show that for every integer $k \geq 2\nu_o(G) - 1$, the modules $J(G)^{(k)}$ and $S/J(G)^{(k)}$ satisfy the Stanley's inequality. We also provide an alternative proof for [9, Theorem 3.4] which states that $\mathrm{depth}(S/J(G)^{(k)}) = n - \nu_o(G) - 1$, for every integer $k \geq 2\nu_o(G) - 1$.

1. Introduction

Let $S = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field $\mathbb{K}$ and suppose that M is a nonzero finitely generated $\mathbb{Z}^n$-graded S-module. Let $u \in M$ be a homogeneous element and $Z \subseteq \{x_1, \ldots, x_n\}$. The $\mathbb{K}$-subspace $u\mathbb{K}[Z]$ generated by all elements uv with $v \in \mathbb{K}[Z]$ is called a *Stanley space* of dimension $|Z|$, if it is a free $\mathbb{K}[Z]$-module. Here, as

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usual, $|Z|$ denotes the number of elements of Z. A decomposition $\mathcal{D}$ of M as a finite direct sum of Stanley spaces is called a *Stanley decomposition* of M. The minimum dimension of a Stanley space in $\mathcal{D}$ is called the *Stanley depth* of $\mathcal{D}$ and is denoted by $\mathrm{sdepth}(\mathcal{D})$. The quantity

$$\mathrm{sdepth}(M) := \max\big\{\mathrm{sdepth}(\mathcal{D}) \mid \mathcal{D} \text{ is a Stanley decomposition of } M\big\}$$

is called the *Stanley depth* of M. As a convention, we set $\mathrm{sdepth}(M) = \infty$, when M is the zero module. We say that a $\mathbb{Z}^n$-graded S-module M satisfies the *Stanley's inequality* if

$$\mathrm{depth}(M) \leq \mathrm{sdepth}(M).$$

In fact, Stanley [20] conjectured that every $\mathbb{Z}^n$-graded S-module satisfies the Stanley's inequality. We refer to [14] for a reader friendly introduction to Stanley depth, and to [7] for a nice survey on this topic.

The Stanley's conjecture has been recently disproved in [5]. The counterexample presented in [5] lives in the category of squarefree monomial ideals. Thus, one can still ask whether the Stanley's inequality holds for non-squarefree monomial ideals. Of particular interest are the high powers of monomial ideals. In other words, we ask following question.

Question 1.1. Let I be a monomial ideal. Is it true that I^k and S/I^k satisfy the Stanley's inequality for every integer $k \gg 0$?

We approached this question for edge ideals in [1], [15] and [17]. The most general results are obtained in [17]. In that paper, we proved that if G is a graph with n vertices and $I(G)$ is its edge ideal, then $S/I(G)^k$ satisfies the Stanley's inequality for every integer $k \geq n-1$ [17, Corollary 2.5]. If moreover G is a non-bipartite graph, or at least one of the connected components of G is a tree with at least one edge, then $I(G)^k$ satisfies the Stanley's inequality for every integer $k \geq n-1$ [17, Corollary 3.6].

Recently, in [16], we studied the powers of cover ideal of bipartite graphs. We proved in [16, Corollary 3.5] that if G is a bipartite graph with cover ideal $J(G)$, then $J(G)^k$ and $S/J(G)^k$ satisfy the Stanley's conjecture for $k \gg 0$. On the other hand, we know from [6, Corollary 2.6] that for every bipartite graph G and every integer $k \geq 1$, we have $J(G)^k = J(G)^{(k)}$, where $J(G)^{(k)}$ denotes the k-th symbolic power of $J(G)$. Hence, [16, Corollary 3.5] essentially says that for any bipartite graph G, the modules $J(G)^{(k)}$ and $S/J(G)^{(k)}$ satisfy the Stanley's inequality for every integer $k \gg 0$. In this regard, we ask an analogue of Question 1.1 for symbolic powers.

Question 1.2. Let I be a squarefree monomial ideal. Is it true that $I^{(k)}$ and $S/I^{(k)}$ satisfy the Stanley's inequality for every integer $k \gg 0$?