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Images of derivations of polynomial algebras with divergence zero ☆

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ABSTRACT

In this paper, we show that the image of a derivation D of $k[x, y]$ with divergence zero is not necessarily a Mathieu subspace, where k is a field of characteristic zero, which gives a negative answer to Question 4.1 proposed by van den Essen, Wright and Zhao in [4]. Note that the 2-dimensional Jacobian conjecture is equivalent to saying that, the image of any derivation D of $k[x, y]$ with divergence zero and $1 \in \operatorname{Im} D$ is a Mathieu subspace.

1. Introduction

Throughout the paper, k stands for a field of characteristic zero. The Jacobian conjecture asserts that a polynomial map $F : k^n \to k^n$ with nonzero constant Jacobian determinant is invertible, see [3] or [1]. It is still open for any $n \geq 2$.

The interest in images of differential operators (including derivations) arose in the last few years, since it was found that the study of them is closely related to the Jacobian conjecture.

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On one hand, Zhao [9] showed that if the image $\mathrm{Im}\Psi := \sum_{i=1}^{n}(\xi_i - \partial_{x_i})P$ of the set of differential operators $\Psi := \{\xi_i - \partial_{x_i}, 1 \leq i \leq n\}$ of the polynomial algebra $P = k[\xi_1, \dots, \xi_n, x_1, \dots, x_n]$ is a Mathieu subspace for all $n \geq 1$, then the Jacobian conjecture holds for all $n \geq 1$. For subsequent investigations on $\mathrm{Im}\Psi$, we refer the reader to [5,7,10] etc. Up to now, $\mathrm{Im}\Psi$ was only verified to be a Mathieu subspace for $n = 1$, and it is still open for any $n \geq 2$.

On the other hand, van den Essen, Wright and Zhao [4] related the study of images of derivations with the Jacobian conjecture; in fact, they proposed the following two questions, and showed that Question 1.2 (a special case of Question 1.1) has an affirmative answer if and only if the 2-dimensional Jacobian conjecture holds. In what follows, we always denote by $A = k[x, y]$ the two-variable polynomial algebra over k.

Question 1.1. [4, Question 4.1] Let D be a k-derivation of $A = k[x, y]$ with $\mathrm{div}D = 0$. Is $\mathrm{Im}D$ a Mathieu subspace of A?

Question 1.2. [4, Question 4.2] Let D be a k-derivation of $A = k[x, y]$ with $\mathrm{div}D = 0$ and $1 \in \mathrm{Im}D$. Is $\mathrm{Im}D$ a Mathieu subspace of A?

The divergence of a k-derivation D of $k[x, y]$ is $\mathrm{div}D := \partial_x(D(x)) + \partial_y(D(y))$. For the definition of Mathieu subspaces, see Definition 2.1 below. The notion of Mathieu subspaces was introduced by Zhao in [10], which is a natural generalization of ideals. For more properties of Mathieu subspaces we refer the reader to [11,7].

In [4], it was showed that the image of any locally finite derivation of $A = k[x, y]$ is a Mathieu subspace. (A k-derivation D is called locally finite if for every $f \in A$ the subspace of A spanned by the elements $D^i(f)(i \geq 1)$ is finite dimensional.)

In this paper, we investigate homogeneous monomial derivations D of $A = k[x, y]$ with $\mathrm{div}D = 0$, and give a negative answer to Question 1.1. In fact, we show that, for a derivation of the form $D = bx^a y^{b-1}\partial_x - ax^{a-1}y^b\partial_y$, $a \geq 1, b \geq 1$, $\mathrm{Im}D$ is a Mathieu subspace if and only if $a = b$ (Theorem 2.6).

2. Images of derivations with divergence zero

We start with definitions concerning Mathieu subspaces and derivations.

Definition 2.1. [10] A subspace M of a commutative k-algebra S is called a Mathieu subspace if any of the following equivalent properties holds:

1. If $f \in S$ is such that $f^m \in M$, $\forall m \geq 1$, then for every $g \in S$, we have $gf^m \in M$, $\forall m \gg 0$ (i.e., there exists some m_g such that $gf^m \in M$, $\forall m \geq m_g$);
2. If $f \in S$ is such that $f^m \in M$, $\forall m \gg 0$, then for every $g \in S$, we have $gf^m \in M$, $\forall m \gg 0$.