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Construction of idempotent 2-cocycles

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ABSTRACT

Let $f : G \times G \longrightarrow L$ be a weak 2-cocycle, where L/K is a finite Galois field extension with Galois group G, and $A_f = (L/K, f)$ be the associated weak crossed product K-algebra. We associate a partition of G to A_f and construct idempotent 2-cocycles of A_f.

1. Introduction

Let L/K be a finite Galois field extension with Galois group G and $L^* := L \setminus \{0\}$. A function $f : G \times G \longrightarrow L^*$ is called normalized 2-cocycle when the following conditions are satisfied:

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$$\begin{cases} f(\sigma,\tau)f(\sigma\tau,\rho) = f^{\sigma}(\tau,\rho)f(\sigma,\tau\rho), & \text{for } \sigma,\tau,\rho \in G \\ f(\sigma,1) = f(1,\sigma) = 1, \quad \text{for } \sigma \in G, \end{cases} \tag{1.1}$$

where l^{σ} means $\sigma(l)$, for $\sigma \in G$ and $l \in L$, and 1 is the unit element of G. If we allow f to take values on L instead of L^*, then we are referring to a weak 2-cocycle. Two weak 2-cocycles f and g are called cohomologous, written $f \sim g$, if there is a function $a : G \longrightarrow L^*$ such that

$$f(\sigma,\tau) = \frac{a(\sigma)a^{\sigma}(\tau)}{a(\sigma\tau)}g(\sigma,\tau), \quad \text{for } \sigma,\tau \in G.$$

Let $M^2(G,L)$ be the set of equivalent classes of the weak 2-cocycles from G to L, which under the pointwise multiplication forms a monoid.

D.E. Haile, R.G. Larson and M.E. Sweedler in [3] studied the weak 2-cocycles and the monoid $M^2(G,L)$, where they introduced a new cohomology theory based on weak 2-cocycles. The resulting cohomology monoids give new invariants even in classical settings as $\mathbb{C}$ and $\mathbb{R}$. Associated to a weak 2-cocycle f there is a K-algebra A_f, called the weak crossed product algebra associated to f. The K-algebra A_f is defined as a K-vector space

$$A_f = \bigoplus_{\sigma \in G} Lu_{\sigma}$$

having as an L-basis the elements u_{σ}, $\sigma \in G$, and multiplication defined by the rules

$$u_{\sigma}l = l^{\sigma}u_{\sigma} \quad \text{and} \quad u_{\sigma}u_{\tau} = f(\sigma,\tau)u_{\sigma\tau},$$

for all $\sigma,\tau \in G$ and $l \in L$. The cocycle condition (1.1) guarantees that A_f is an associative K-algebra with unit element u_1, that we denote also by 1. It is easy to see that, for f,g two weak 2-cocycles, $f \backsim g$ if and only if there is a K-algebra isomorphism $\phi : A_f \to A_g$ such that $\phi|_L = 1_L$. Let $H(f) = \{\sigma \in G : f(\sigma,\sigma^{-1}) \neq 0\}$. It was shown in ([3], Section 10) that $H(f)$ is a subgroup of G, called the inertial subgroup of f, and

$$A_f = B \oplus J_f, \tag{1.2}$$

where

$$B = \bigoplus_{\sigma \in H(f)} Lu_{\sigma} \quad \text{and} \quad J_f = \bigoplus_{\sigma \notin H(f)} Lu_{\sigma}.$$

The algebra B is a central simple L^H-algebra, where L^H is the fixed field of $H(f)$, and J_f is the Jacobson radical of A_f. In other words the relation (1.2) gives the Wedderburn splitting of A_f. In ([3], Section 7) it was shown that