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Construction of flows of finite-dimensional algebras

M. Ladra [a], U.A. Rozikov [b,*]

[a] *Department of Mathematics, University of Santiago de Compostela, 15782, Spain*
[b] *Institute of Mathematics, 29, Do'rmon Yo'li str., 100125, Tashkent, Uzbekistan*

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A B S T R A C T

Recently, we introduced the notion of flow (depending on time) of finite-dimensional algebras. A flow of algebras (FA) is a particular case of a continuous-time dynamical system whose states are finite-dimensional algebras with (cubic) matrices of structural constants satisfying an analogue of the Kolmogorov–Chapman equation (KCE). Since there are several kinds of multiplications between cubic matrices one has fix a multiplication first and then consider the KCE with respect to the fixed multiplication. The existence of a solution for the KCE provides the existence of an FA. In this paper our aim is to find sufficient conditions on the multiplications under which the corresponding KCE has a solution. Mainly our conditions are given on the algebra of cubic matrices (ACM) considered with respect to a fixed multiplication of cubic matrices. Under some assumptions on the ACM (e.g. power associative, unital, associative, commutative) we describe a wide class of FAs, which contain algebras of arbitrary finite dimension. In particular, adapting the theory of continuous-time Markov processes, we construct a class of FAs given by the matrix exponent of cubic matrices. Moreover, we remarkably extend the set of FAs given with respect to the Maksimov's multiplications of our paper [8]. For several FAs we study the time-dependent behavior (dynamics) of the algebras. We derive a system of differential equations for FAs.

* Corresponding author.
E-mail addresses: manuel.ladra@usc.es (M. Ladra), rozikovu@yandex.ru (U.A. Rozikov).

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1. Introduction

It is known that (see e.g. [16]) if each element of a family (depending on time) of matrices satisfying the Kolmogorov–Chapman equation (KCE) is stochastic, then it generates a Markov process. But what kind of processes or dynamical systems can be generated by a family of non-stochastic matrices satisfying KCE? Depending on the matrices, it can be a non-Markov process [4], a deformation [12], etc. Other motivations of consideration of non-stochastic solutions of KCE are given in recent papers [2,11,13–15]. These papers are devoted to study some chains of evolution algebras. In each of these papers the matrices of structural constants (time-dependent on the pair (s, t)) are square or rectangular and satisfy the KCE. In other words, a chain of evolution algebras is a continuous-time dynamical system which in any fixed time is an evolution algebra.

In [3], [7] some cubic stochastic matrices are used (as matrices of structural constants) to investigate algebras and dynamical systems of bisexual populations.

In [8] we generalized the notion of chain of evolution algebras (given for algebras with *rectangular* matrices) to a notion of flow of arbitrary finite-dimensional algebras (i.e. their matrices of structural constants are *cubic* matrices). In this paper we continue our investigations of flows of algebras (FAs).

The paper is organized as follows. In Section 2 we give the main definitions related to algebras of cubic matrices, several kinds of multiplications of cubic matrices and FAs. Note that an FA is defined by a family (depending on time) of cubic matrices of structural constants, which satisfy an analogue of KCE. Since there are several types of multiplication of cubic matrices, one has to fix a multiplication, say μ, first and then consider the KCE with respect to this multiplication. In Section 3 we find some conditions on μ under which the KCE has at least one solution. For several multiplications we give a wide class of solution of KCE, i.e. a wide class of FAs. For the multiplications of Maksimov [9] we extend the class of FAs given in [8]. Section 4 contains some differential equations for FAs. We study the time-dependent behavior of the different examples of flows constructed in this paper.

2. Definitions

2.1. Algebras of cubic matrices

Given a field F, any finite-dimensional algebra $\mathcal{A}$ can be specified up to isomorphism by giving its dimension (say m), and specifying m^3 structural constants c_{ijk}, which are scalars in F. These structure constants determine the multiplication in $\mathcal{A}$ via the following rule:

$$e_i e_j = \sum_{k=1}^{m} c_{ijk} e_k,$$