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Asymptotic Hilbert–Kunz multiplicity

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ABSTRACT

For a pair (M, I), where M is finitely generated graded module over a standard graded ring R of dimension $d \geq 2$, and I is a graded ideal with $\ell(R/I) < \infty$ and generated by elements of the same degree, we prove that $\lim_{q\to\infty} e_1(M, I^{[q]})/q^d$ exists, where $e_1(M, I^{[q]})$ denotes the first coefficient of the Hilbert–Samuel polynomial of $(M, I^{[q]})$.
We use this to get an expression for $\lim_{k\to\infty}[e_{HK}(M, I^k) - e_0(M, I^k)/d!]/k^{d-1}$, where e_{HK} denotes the Hilbert–Kunz multiplicity. In particular, if dim $M = d$ then we deduce that the difference $e_{HK}(M, I^k) - e(M, I^k)/d!$ grows at least as a fixed positive multiple of k^{d-1} as $k \to \infty$.
This is proved using 'renormalized' HK density functions.

1. Introduction

Let R be a Noetherian ring of prime characteristic $p > 0$ and of dimension d and let $I \subseteq R$ be an ideal of finite colength. Then we recall that the Hilbert–Kunz multiplicity of R with respect to I is defined as

$$e_{HK}(R, I) = \lim_{n\to\infty} \frac{\ell(R/I^{[p^n]})}{p^{nd}},$$

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where $I^{[p^n]}$ = (n-th Frobenius power of I) = (the ideal generated by p^n-th powers of elements of I). This is an ideal of finite colength and $\ell(R/I^{[p^n]})$ denotes the length of the R-module $R/I^{[p^n]}$. Existence of the limit was proved by Monsky [5]. This invariant has been extensively studied, over the years (see the survey article [3]). Since various standard techniques, used for studying multiplicities, are not applicable for the invariant e_{HK}, it has been difficult to handle (even in the graded case).

In this paper, we study the behaviour of $e_{HK}(M, I^k)$, as a function of k. This problem was first studied by Watanabe–Yoshida in [7], for a Noetherian local ring $(R, \mathbf{m})$ of dimension $d \geq 2$ and an $\mathbf{m}$-primary ideal I. In particular, in [7] it is shown that

$$\frac{e_0(R, I^k)}{d!} \leq e_{HK}(R, I^k) \leq \frac{\binom{k+d-1}{d}}{k^d} e_0(R, I^k),$$

and as a corollary they get

$$e_{HK}(R, I^k) = \frac{e_0(R, I)}{d!} k^d + o(k^d).$$

Later Hanes in [2] improved this as follows

$$\ell(R/I^{[q]k}) = \left[\frac{e_0(R, I)}{d!} k^d + O(k^{d-1})\right] q^d.$$

We introduce here another invariant $E_1(M, I)$, for a pair (M, I) where M is a graded module over a standard graded ring R of dimension $d \geq 2$ (over a perfect field of char $p > 0$), and I is a homogeneous ideal generated by homogeneous elements of a fixed degree (*e.g.*, the graded maximal ideal).

We define this as

$$E_1(M, I) := \lim_{q \to \infty} e_1(M, I^{[q]})/q^d,$$

where $e_1(M, J)$ denotes the first coefficient of the Hilbert–Samuel polynomial of M with respect to J. It is well known that the $0th$ coefficient of Hilbert polynomial (which is the Hilbert–Samuel multiplicity) satisfies $e_0(M, I^{[q]}) = q^d e_0(M, I)$ and therefore $\lim_{q\to\infty} e_0(M, I^{[q]})/q^d$ exists, but we did not know the similar existence of a limit for the first coefficient of the Hilbert–Samuel polynomial. This new invariant seems rather mysterious, but it relates the asymptotic growth of $e_{HK}(M, I^k)$ with $k^d e_0(M, I)$, and also satisfies an additive formula. More precisely we prove the following results in this paper (see Theorem 3.6, Remark 3.7, Proposition 3.8 and Lemma 3.10).

Theorem. *Let R be a standard graded ring of dimension $d \geq 2$ over a perfect field K of* char $p > 0$*, and let $I \subset R$ be a homogeneous ideal of finite colength, which has a set of generators of the same degree. Let M be a finitely generated graded R-module. Then*