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Finitely generated algebras defined by homogeneous quadratic monomial relations and their underlying monoids II ☆

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A B S T R A C T

We continue our investigations on algebras R over a field K with generators $x_1, x_2, \ldots, x_n$ subject to $\binom{n}{2}$ quadratic relations of the form $x_i x_j = x_k x_l$ with $(i,j) \neq (k,l)$ and, moreover, every monomial $x_i x_j$ appears at most once in one of the defining relations. If these relations are non-degenerate then it is shown that the underlying monoid S contains an abelian submonoid $A = \langle s^N \mid s \in S \rangle$, that is finitely generated and that $S = \bigcup_{f \in F} fA = \bigcup_{f \in F} Af$ for some finite subset F of S. So, $R = K[S]$ is a finite module over the Noetherian commutative algebra $K[A]$; in particular R is a Noetherian algebra that satisfies a polynomial identity. Well-known examples of such monoids are the monoids of I-type that correspond to non-degenerate set-theoretical solutions of the Yang–Baxter equation. We show that S is of I-type if and only if S is cancellative and satisfies the cyclic condition. Furthermore, if S satisfies the cyclic condition, then S is cancellative if and only of $K[S]$ is a prime ring. Moreover, in this case, one can replace the monoid A by a finitely generated submonoid A' such that $fA' = A'f$, for each $f \in F$; in particular $R = K[S]$ is a normalizing extension of $K[A']$ and thus the prime ideals of $K[S]$ are determined by the prime ideals of $K[A']$. These investigations are a continuation and generalization of earlier results of Cedó, Gateva–Ivanova,

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Jespers and Okniński in the case the defining relations are square free.

1. Introduction

We continue the investigations started in [18] on algebras over a field K with generators $x_1, x_2, \dots, x_n$ (with $n \geqslant 2$) subject to $\binom{n}{2}$ monomial relations of degree two. More precisely, such an algebra is the semigroup algebra $K[S]$ of a monoid, called a quadratic monoid, $S = \langle x_1, x_2, \dots, x_n \mid R\rangle$, where R is a finite set consisting of $\binom{n}{2}$ relations that are of the type $x_i x_j = x_k x_l$ with $(i, j) \neq (k, l)$ and, moreover, every word $x_i x_j$ appears at most once in all the relations. These finitely presented algebras have in recent years attracted a lot of attention as, under some non-degenerate assumption on the relations, they provide a large class of Noetherian algebras that satisfy a polynomial identity and that have finite Gelfand–Kirillov dimension, actually this dimension is at most n (Theorem 3.2 in [18]). This was first proved in [13] in case S is a non-degenerate monoid of skew type (see [13,17]), that is, the defining relations R of the non-degenerate quadratic monoid S also are square free, i.e. if $x_i x_j = x_k x_l$ is such a relation then $i \neq j$ and $k \neq l$. For some special classes of such non-degenerate quadratic monoids (called monoids of I-tye), these algebras satisfy strong homological properties, such as being Koszul, Auslander–Gorenstein and Cohen–Macauly. Furthermore, the monoids M of I-type yield set-theoretic solutions of the Yang–Baxter equation. It is here that lies some of the origins of the monoids under consideration. There is an extensive literature on the properties of the algebras $K[M]$ and on related investigations (see for example [3–5,7–9,12,15,17,16,21,22]).

In order to state the main results obtained in this paper, we recall some terminology. So, let $S = \langle x_1, x_2, \dots, x_n \mid R\rangle$ be a quadratic monoid and put $X = \{x_1, x_2, \dots, x_n\}$. Since the defining relations are homogeneous, one can define the length $|w|$ of $1 \neq w \in S$ as the length of any decomposition of w as a product of elements of X; by definition $|1| = 0$. Note that there are precisely n words of length 2 not showing up in any of the defining relations, or equivalently, there are precisely $n^2 - n$ such words that show up in one of the defining relations. Hence, one associates to S the bijective map

$$r : X \times X \to X \times X,$$

defined by $r(x_i, x_j) = (x_k, x_l)$ if $x_i x_j = x_k x_l$ is a defining relation for S, and $r(x_i, x_j) = (x_i, x_j)$ if $x_i x_j$ does not occur in any defining relation. Note that implicitly we also define $r(x_k, x_l) = (x_i, x_j)$, if $x_i x_j = x_k x_l$ is a defining relation for S, and hence r is an involution. For every $x \in X$, let

$$f_x : X \to X \quad \text{and} \quad g_x : X \to X$$