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Endomorphism algebras for a class of negative Calabi–Yau categories

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ABSTRACT

We consider an orbit category of the bounded derived category of a path algebra of type A_n which can be viewed as a $-(m+1)$-cluster category, for $m \geqslant 1$. In particular, we give a characterisation of those maximal m-rigid objects whose endomorphism algebras are connected, and then use it to explicitly study these algebras. Specifically, we give a full description of them in terms of quivers and relations, and relate them with (higher) cluster-tilted algebras of type A. As a by-product, we introduce a larger class of algebras, called *tiling algebras*.

Introduction

In [5], and independently in [6] in type A, the authors introduced cluster categories as a categorical model of cluster algebras. These categories led to the development of

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so-called cluster-tilting theory, an important generalisation of classical tilting theory, and are therefore of central importance in representation theory.

The cluster category of an acyclic quiver Q is defined to be the $\tau^{-1}\Sigma$-orbit category of the bounded derived category $\mathsf{D}^b(\mathbf{k}Q)$ of the corresponding path algebra $\mathbf{k}Q$, where τ is the Auslander–Reiten translate and Σ is the suspension functor in $\mathsf{D}^b(\mathbf{k}Q)$. In [17], Thomas introduced a generalisation of cluster categories, the so-called m-cluster categories, for $m \geqslant 1$. These are the $\tau^{-1}\Sigma^m$-orbit categories of $\mathsf{D}^b(\mathbf{k}Q)$. A key property of these categories, which inspired several authors to study cluster-tilting theory in a more general set-up, is that they are $(m+1)$-Calabi–Yau triangulated categories.

In the present article, we will consider the orbit categories $\mathsf{B}_m(A_n)$ of $\mathsf{D}^b(\mathbf{k}A_n)$ by $\tau\Sigma^{m+1}$, for $m \geqslant 1$, where $\mathbf{k}A_n$ is a path algebra of type A_n. These categories can be viewed as $-(m+1)$-cluster categories and $(-m)$-Calabi–Yau (Σ^{-m} is the Serre functor). In particular, they can be considered to be of negative Calabi–Yau 'dimension'. Further reasons to support this idea can be found in [8] and [9].

The main interest in m-cluster categories, and other positive CY-triangulated categories, has arisen from the nice homological and combinatorial properties of m-cluster-tilting objects and their corresponding endomorphism algebras. In the acyclic case, m-cluster-tilting objects coincide with maximal m-rigid objects: objects which are maximal with respect to the property that $\mathsf{Ext}^i(t, t') = 0$, for every pair of summands t, t' and for all $1 \leqslant i \leqslant m$.

In this article, we will study a subclass of the maximal m-rigid objects of $\mathsf{B}_m(A_n)$, namely those whose endomorphism algebras are connected. We note that, when $m = 1$, this is the whole class of maximal m-rigid objects.

Cluster-tilting theory has also led to an interest in associating combinatorial models to triangulated categories, in particular the (higher-) cluster categories. These models facilitate the provision of simple characterisations of several representation-theoretic objects, which are more tractable than the objects themselves. For instance, m-cluster-tilting objects in type A_n can be simply described via $(m+2)$-angulations of an $(m(n+1)+2)$-gon (see [4]). Note that when at least one of the $(m+2)$-gons in an $(m+2)$-angulation has two disjoint boundary segments, the corresponding m-cluster-tilted algebra is disconnected.

We will also make use of a combinatorial model for $\mathsf{B}_m(A_n)$. This model, in which indecomposable objects correspond to the '$(m+1)$-diagonals' of an $((m+1)(n+1)-2)$-gon was introduced in [8]. We note that, in the case when $m = 1$, maximal rigid objects were characterised in [7] using a different combinatorial model, in terms of oriented diagonals in an n-gon. The characterisation (Theorem 2.10) we present in this article is not only more general, it also has a simpler description, which enables us to develop a deeper understanding of the corresponding endomorphism algebras.

The collection of $(m+1)$-diagonals (viewed inside the polygon) corresponding to a maximal m-rigid object in $\mathsf{B}_m(A_n)$ can 'admit crossings' or 'contain regions bounded by disjoint boundary segments'; in which case the associated endomorphism algebra is disconnected. The behaviour of such collections of $(m+1)$-diagonals is not as neat, and so we will restrict to the connected case. The connected endomorphism algebras can be