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Deformations of complexes for finite dimensional algebras

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ABSTRACT

Let k be a field and let Λ be a finite dimensional k-algebra. We prove that every bounded complex $V^\bullet$ of finitely generated Λ-modules has a well-defined versal deformation ring $R(\Lambda, V^\bullet)$ which is a complete local commutative Noetherian k-algebra with residue field k. We also prove that nice two-sided tilting complexes between Λ and another finite dimensional k-algebra Γ preserve these versal deformation rings. Additionally, we investigate stable equivalences of Morita type between self-injective algebras in this context. We apply these results to the derived equivalence classes of the members of a particular family of algebras of dihedral type that were introduced by Erdmann and shown by Holm to be not derived equivalent to any block of a group algebra.

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1. Introduction

The main objective of the theory of deformations of algebraic objects, such as modules or group representations, is to study the behavior of these objects under perturbations. Suppose $\mathcal{O}$ is a complete local commutative Noetherian ring with residue field k, $\Lambda_{\mathcal{O}}$ is an $\mathcal{O}$-algebra and $\Lambda = k \otimes_{\mathcal{O}} \Lambda_{\mathcal{O}}$. If V is a Λ-module of finite k-dimension, the deformations of V are defined to be isomorphism classes of lifts of V over complete local commutative Noetherian $\mathcal{O}$-algebras R with residue field k. Here a lift of V over R is an $R \otimes_{\mathcal{O}} \Lambda_{\mathcal{O}}$-module M that is free over R, together with a Λ-module isomorphism $\phi : k \otimes_R M \to V$.

Lifts and deformations of this kind were studied by Green in [18] in the 1950's in the case when $\mathcal{O}$ is a ring of p-adic integers, for some prime number p, and $\Lambda_{\mathcal{O}}$ is the group algebra of a group G over $\mathcal{O}$. Green's work inspired Auslander, Ding and Solberg in [1] to consider more general $\mathcal{O}$-algebras $\Lambda_{\mathcal{O}}$ and more general lifting problems. In the 1970's Laudal developed a theory of formal moduli of algebraic structures, and he used Massey products to describe deformations of k-algebras and their modules over complete local commutative Artinian k-algebras with residue field k (see [23] and its references). In the 1980's Mazur developed a theory of deformations of group representations to systematically study p-adic lifts of representations of profinite Galois groups over finite fields of characteristic p (see [25,26]). Both Laudal and Mazur used Schlessinger's criteria in [32] for the pro-representability of functors of Artinian rings. One advantage of Mazur's approach is that he uses a continuous deformation functor, which allows him to include the deformations over arbitrary complete local commutative Noetherian $\mathcal{O}$-algebras with residue field k directly in his functorial description and not just as inverse limits. In particular, Mazur proved in [25] that a finite dimensional Galois representation over a finite field always has a versal deformation ring, and in the case when the representation is absolutely irreducible that this versal deformation ring is universal. In [9], Mazur's approach was used by the authors to study deformation rings and deformations of modules for arbitrary finite dimensional k-algebras Λ when $\mathcal{O} = k$, and to provide additional structure theorems in the case when Λ is self-injective or Frobenius.

Let now k be a field of arbitrary characteristic, let $\mathcal{O} = k$, and let $\Lambda = \Lambda_{\mathcal{O}}$ be a finite dimensional k-algebra. Our first goal is to generalize the deformation theory for finitely generated Λ-modules in [9] to bounded complexes of finitely generated Λ-modules. We accomplish this goal in Section 2. Many of our techniques are based on the generalization of Mazur's deformation theory to bounded complexes of Galois representations in [5,6].

More precisely, let $D^-(\Lambda)$ be the derived category of bounded above cochain complexes of pseudocompact Λ-modules (see Section 2.1 for a review of pseudocompact rings and modules). The following is our first main result (see Theorem 2.1.12 for a more precise statement):

Theorem 1.1. *Let $V^\bullet$ be an object of $D^-(\Lambda)$ such that $V^\bullet$ only has finitely many non-zero cohomology groups, all of which have finite k-dimension. Then $V^\bullet$ always has a versal*