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Direct and central products of localities

Ellen Henke

Institute of Mathematics, University of Aberdeen, Fraser Noble Building, Aberdeen AB24 3UE, UK

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ABSTRACT

We develop a theory of direct and central products of partial groups and localities.

1. Introduction

Partial groups and localities were introduced by Chermak [6], in the context of his proof of the existence and uniqueness of centric linking systems. Roughly speaking, a partial group is a set $\mathcal{L}$ together with a product which is only defined on certain words in $\mathcal{L}$, and an inversion map $\mathcal{L} \to \mathcal{L}$ which is an involutory bijection, subject to certain axioms. A locality is a partial group equipped with some extra structure which makes it possible to define the fusion system of a locality. Essentially, localities are "the same" as the transporter systems of Oliver and Ventura [11]; see the appendix to [6]. As centric

E-mail address: ellen.henke@abdn.ac.uk.

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linking systems are special cases of transporter systems, the existence of centric linking systems implies that there is a locality attached to every fusion system. It is work in progress of Chermak and the author of this paper to build a local theory of localities similar to the local theory of fusion systems as developed by Aschbacher [2], [3] based on earlier work of many other authors. The results we prove in this paper fit into this program.

For fusion systems, a relatively canonical definition of an external direct product was already introduced by Broto, Levi and Oliver [5]. Building on this definition, Aschbacher [3] introduced central products of fusion systems. In this paper, we develop a theory of direct and central products of partial groups and localities. Most of our definitions are again quite canonical. After some preliminaries, we introduce in Section 4 direct products of partial groups and prove basic properties of these. This allows us in Section 5 to define external direct and central products of localities. In Section 6 we introduce internal direct and central products of partial groups and localities, and we prove results relating them to their external counterparts.

Of special interest are localities corresponding to centric linking systems or, more generally, linking localities as introduced in [9]. We prove that an external or internal direct product of two localities is a linking locality if and only if the two localities we started with are linking localities. A similar result holds for central products. The reader is referred to Lemma 5.7, Lemma 5.9(b) and Lemma 6.11 for the precise statements of the results.

Given a linking locality over a saturated fusion system $\mathcal{F}$, it is recent work of Chermak and the author of this paper [8] to prove that there is a one-to-one correspondence between the normal subsystems of $\mathcal{F}$ and the partial normal subgroups of the locality. A significant part of the theory developed in this paper is needed in the proof. In particular, at the end we prove Proposition 6.12 with this application in mind.

Throughout, p is always a prime. We will use the right hand notation for maps.

2. Background on fusion systems

2.1. Some notation and terminology

We refer the reader to [4, Part I] for background on fusion systems, but we recall some notation and terminology here.

Let $\mathcal{F}$ be a fusion system over S. A subgroup $R \leq S$ is *normal* in $\mathcal{F}$ if $R \trianglelefteq S$ and, for all $P, Q \leq S$, every morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ extends to a morphism $\hat{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(PR, QR)$ with $R\hat{\varphi} = R$. Similarly, a subgroup $R \leq S$ is *central* in $\mathcal{F}$ if $R \trianglelefteq S$ and, for all $P, Q \leq S$, every morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ extends to a morphism $\hat{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(PR, QR)$ with $\hat{\varphi}|_R = \mathrm{id}_R$. It follows from these definitions that there exists a largest normal subgroup of $\mathcal{F}$, which is denoted by $O_p(\mathcal{F})$, and a largest central subgroup of $\mathcal{F}$, which is denoted by $Z(\mathcal{F})$. A subgroup R of $\mathcal{F}$ is called *strongly closed* if $X\varphi \leq R$ for every $X \leq R$ and every $\varphi \in \mathrm{Hom}_{\mathcal{F}}(X, S)$. Note that every strongly closed subgroup of $\mathcal{F}$ is normal in S.