Journal of Algebra 491 (2017) 190–206

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

# Square-free class sizes in products of groups

M.J. Felipe, A. Martínez-Pastor *, V.M. Ortiz-Sotomayor

*Instituto Universitario de Matemática Pura y Aplicada (IUMPA), Universitat Politècnica de València, Camino de Vera, s/n, 46022, Valencia, Spain*

A R T I C L E   I N F O

*Article history:*
Received 23 March 2017
Available online 18 August 2017
Communicated by Gernot Stroth

*MSC:*
20D10
20D40
20E45

*Keywords:*
Finite groups
Soluble groups
Products of subgroups
Conjugacy classes

A B S T R A C T

We obtain some structural properties of a factorised group $G = AB$, given that the conjugacy class sizes of certain elements in $A \cup B$ are not divisible by $p^2$, for some prime $p$. The case when $G = AB$ is a mutually permutable product is especially considered.



## 1. Introduction

All groups considered throughout this paper are finite. Over the last years, many authors have investigated the influence of conjugacy class sizes on the structure of finite groups. In the meantime, numerous studies in the framework of group theory have focused

* Corresponding author.
*E-mail addresses:* mfelipe@mat.upv.es (M.J. Felipe), anamarti@mat.upv.es (A. Martínez-Pastor), vicorso@doctor.upv.es (V.M. Ortiz-Sotomayor).

http://dx.doi.org/10.1016/j.jalgebra.2017.08.007

in factorised groups. In this setting, a central question is how the structure of the factors affects the structure of the whole group, in particular when they are connected by certain permutability properties. The purpose of this paper is to show new achievements which combine both current perspectives in finite groups. More precisely, our aim is to get some information about a factorised group, provided that the conjugacy class sizes of some elements of its factors are square-free.

The earlier starting point of our investigation can be traced back to the paper of Chillag and Herzog ([5]), where the structure of a group in which all elements have square-free conjugacy class sizes was first analysed. Next, in [6], Cossey and Wang localised one of the main theorems in [5] for a fixed prime $p$, that is, they considered conjugacy class sizes not divisible by $p^2$, for certain prime $p$. Later on, this study was improved by Li in [12], and by Liu, Wang, and Wei in [13], by replacing conditions on all conjugacy classes by those referring only to conjugacy classes of either $p$-regular elements or prime power order elements, using the classification theorem of finite simple groups (CFSG). These authors also first obtained some preliminary results in factorised groups. This research was extended in 2012 by Ballester-Bolinches, Cossey and Li in [2], through mutually permutable products. More recently, in 2014, Qian and Wang ([14]) have gone a step further by considering just conjugacy class sizes of $p$-regular elements of prime power order (although not in factorised groups).

In the context of factorised groups, and aiming to obtain criteria for products of supersoluble subgroups to be supersoluble, several authors have considered products in which certain subgroups of the factors permute (see [3] for a detailed account). In this scene, we are interested in *mutually permutable products*, factorised groups $G = AB$ such that the subgroups $A$ and $B$ are mutually permutable, i.e., $A$ permutes with every subgroup of $B$ and $B$ permutes with every subgroup of $A$ (see also [4]). Obviously, if $A$ and $B$ are normal in $G$, then they are mutually permutable.

We recall that, for a group $G$, the set $x^G = \{g^{-1}xg \,:\, g \in G\}$ is the *conjugacy class* of the element $x \in G$, and $\left|x^G\right|$ denotes the *conjugacy class size* of $x$. If $p$ is a prime number, we say that $x \in G$ is a *p-regular* element if its order is not divisible by $p$, and that it is a *p-element* if its order is a power of $p$. Moreover, if $n$ is an integer, let $n_p$ denote the highest power of $p$ dividing $n$. The $m$th group of order $n$ in the `SmallGroups` library [8] of GAP will be identified by $n\#m$. The remainder notation is standard and is taken mainly from [7]. We also refer to this book for details about classes of groups.

In this paper, motivated by the above development, at first we focus on the case of $p$-groups, extending for factorised groups the well-known Knoche's theorem (see [11]).

**Theorem A.** *Let $p$ be a prime number and let $P = AB$ be a $p$-group such that $p^2$ does not divide $\left|x^P\right|$ for all $x \in A \cup B$. Then $P' \leqslant \Phi(P) \leqslant \mathrm{Z}(P)$, $P'$ is elementary abelian and $|P'| \le p^2$.*

Our next goal in the paper is to prove the following theorem, regarding mutually permutable products.