Journal of Algebra 491 (2017) 219–240

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

The McDonald theorem in positive characteristic

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ARTICLE INFO

Article history:
Received 3 December 2015
Available online 18 August 2017
Communicated by Steven Dale Cutkosky

MSC:
primary 13F25
secondary 13J05, 12J25

Keywords:
Fractional power series
Puiseux expansions
Algebraically closed
Generalized power series

ABSTRACT

J. McDonald proved that a polynomial with coefficients polynomials in characteristic zero has a root which is a fractional power series with support in some strongly convex polyhedral cone. In this paper we present the analogue of this result in positive characteristic.

1. Introduction

Let $\mathbb{C}$ be the field of complex numbers. The Newton–Puiseux theorem states that the algebraic closure of the field of power series $\mathbb{C}((x))$ is the field of Puiseux series. That is the field formed by the union of all the fields $\mathbb{C}((x^{1/d}))$ where d is a positive integer. The theorem was first known by Newton. Later, Puiseux [21] gave an analytic proof. In fact,

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http://dx.doi.org/10.1016/j.jalgebra.2017.08.006

the theorem holds for any algebraically closed field of characteristic zero. A proof of this theorem can be given constructively by a classical algorithm of Newton (see for example [8] or [10] for a formal presentation).

Just considering Puiseux series is not enough to describe an algebraic closure of the field of power series in more than one variable. Let $\mathbb{k}$ be an algebraically closed field of characteristic zero. The Abhyankar–Jung theorem asserts that the roots of a polynomial with coefficients in the ring of formal power series $\mathbb{k}[[x_1, ..., x_n]]$ whose discriminant is a normal crossing are still Puiseux power series in several variables. This theorem was proven by Jung [15] for $n = 2$ and $\mathbb{k} = \mathbb{C}$ and by Abhyankar [1] in the general case. This result provides a local uniformization of the so called quasi-ordinary surface singularities and plays an important role in the resolution of singularities of surfaces (see for example [28] and [29]).

When $\mathbb{k}$ is an algebraically closed field of characteristic $p \neq 0$, F.J. Rayner describes algebraically closed fields of series with support in a so-called field-family [22]. As a consequence of his main result he shows that the set of all formal power series of the form $\sum_{i\in A} a_i t^i$ where $a_i \in \mathbb{k}$, A is a well-ordered set, $i = m_i/dp^{r_i}$, $m_i \in \mathbb{Z}$, $r_i \in \mathbb{Z}$, $d \in \mathbb{Z}_{>0}$ forms an algebraically closed field. In characteristic zero Rayner shows that the exponents can be taken with bounded denominators [23].

In [24] P. Ribenboim considered generalized power series with exponents in a subtotally ordered torsion-free group. P. Ribenboim extends Rayner's result to perfect fields of positive characteristic. In this case the coefficients of the generalized power series belong to some finite extension of the base field.

J. McDonald [18] showed that a polynomial with coefficients in the ring of polynomials $\mathbb{k}[x_1, ..., x_n]$ where $\mathbb{k}$ is a field of characteristic zero has a root which is a Puiseux series with support contained in some strongly convex rational cone. P.D. González Pérez [13] has also generalized McDonald's result by treating the case of holomorphic coefficients. P.D. González Pérez showed that a polynomial can be decomposed in the ring of Puiseux series with support in some strongly convex polyhedral cone. He describes these cones in terms of the Newton polytope of the discriminant. Moreover in [27] M.J. Soto and J.L. Vicente have taken coefficients in the ring of power series in several variables. In [3] F. Aroca extends these results in the case of complex analytic sets of any codimension. A proof of McDonald's theorem using the notion of the amoeba of a polynomial is given in [5]. If $\mathbb{k}$ is not an algebraically closed field, then the above theorems can be generalized by taking series with coefficients in the union over all finite extensions of $\mathbb{k}$ (see for example [20], [24] and [26]).

Suppose that $\mathbb{k}$ is a field of characteristic $p > 0$. In this case McDonald noted that his result is not true because of the existence of fractional power series solutions which have supports containing limit points (see also [2] and [11]). In fact, there is a good understanding when $n = 1$, Kedlaya [16] describes an algebraic closure of $\mathbb{k}((x))$ where $\mathbb{k}$ is a perfect field of positive characteristic (see also Ali Benhissi's paper [6]).

In [4] F. Aroca and G. Ilardi introduce, in the case of characteristic zero, a family of algebraically closed fields of series with support in some strongly convex polyhedral