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Box moves on Littlewood–Richardson tableaux and an application to invariant subspace varieties

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ABSTRACT

In his 1951 book *"Infinite Abelian Groups"*, Kaplansky gives a combinatorial characterization of the isomorphism types of embeddings of a cyclic subgroup in a finite abelian group. In this paper we first use partial maps on Littlewood–Richardson tableaux to generalize this result to finite direct sums of such embeddings. Our main interest is an application to invariant subspaces of nilpotent linear operators. We develop a criterion to decide if two irreducible components in the representation space are in the boundary partial order.

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1. Introduction

Let Λ be a discrete valuation ring with maximal ideal generator p and radical factor field k.

We study the category of all embeddings $(A \subset B)$ of a submodule A in a finite length Λ-module B. Two examples are of particular interest: If $\Lambda = \widehat{\mathbb{Z}_{(p)}}$ is the completion of the ring of p-adic integers, then we are dealing with embeddings of a subgroup in a finite abelian p-group. Their classification, up to isomorphy, is the well-known Birkhoff Problem [1]. On the other hand, if $\Lambda = k[[T]]$ is the power series ring, then an embedding $(A \subset B)$ consists of a nilpotent linear operator B and an invariant subspace A.

In general, any finite length Λ-module A is isomorphic to a direct sum

$$A \cong \Lambda/(p^{\alpha_1}) \oplus \Lambda/(p^{\alpha_2}) \oplus \ldots \oplus \Lambda/(p^{\alpha_n}),$$

where $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$. Therefore, there is a bijection $(A \mapsto \alpha = (\alpha_1, \ldots, \alpha_n))$ between the set of isomorphism classes of finite length Λ-modules and the set of partitions. The partition α is the *type* of A and we write $A \cong N_\alpha(\Lambda) = N_\alpha$.

It is natural to associate with an embedding $(A \subset B)$ of finite length Λ-modules the triple of partitions (α, β, γ), where α, β, γ are the types of A, B and $C = B/A$, respectively.

We call an embedding $(A \subset B)$ *cyclic* provided the submodule A is cyclic as a Λ-module, that is, A is either indecomposable or zero. Cyclic embeddings have been classified, up to isomorphy, by Kaplansky [3, Theorem 25] in terms of the "height sequence" or "Ulm sequence" of the submodule generator. The first aim in this paper is to derive a simple combinatorial description of the isomorphism types of direct sums of cyclic embeddings. They are given in terms of partial maps on Littlewood–Richardson tableaux (Theorem 2.4).

The main goal of the paper is to shed light on the geometry of the representation space of invariant subspace varieties. Suppose that Λ is the power series ring $k[[T]]$ with coefficients in an algebraically closed field k. The embeddings corresponding to partition type (α, β, γ) form a constructible subset $\mathbb{V}^\beta_{\alpha,\gamma} = \{f : N_\alpha \hookrightarrow N_\beta : \operatorname{Cok} f \cong N_\gamma\}$ of the affine variety of all k-linear homomorphisms $\operatorname{Hom}_k(A, B)$. By the Theorem of Green and Klein [4], the variety is non-empty if and only if there exists a Littlewood–Richardson tableau (LR-tableau) of type (α, β, γ). More precisely, we can assign to each embedding a tableau; by $\mathbb{V}_\Gamma$ we denote the subset of $\mathbb{V}^\beta_{\alpha,\gamma}$ of all embeddings with tableau Γ. Then

$$\mathbb{V}^\beta_{\alpha,\gamma} = \dot{\bigcup_\Gamma} \mathbb{V}_\Gamma,$$

where the union is indexed by the LR-tableaux of type (α, β, γ). The closures (in the Zariski topology) $\overline{\mathbb{V}}_\Gamma$ form the irreducible components of $\mathbb{V}^\beta_{\alpha,\gamma}$, see Section 2.2 and [9] for details.