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Lower bounds on projective levels of complexes

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For an associative ring R, the projective level of a complex F is the smallest number of mapping cones needed to build F from projective R-modules. We establish lower bounds for the projective level of F in terms of the vanishing of homology of F. We then use these bounds to derive a new version of The New Intersection Theorem for level when R is a commutative Noetherian local ring.

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Introduction

Let R be an associative ring and F a complex of left R-modules. The projective level of F, denoted by $\operatorname{level}^{\mathcal{P}}(F)$, is the smallest number of steps needed to assemble some projective resolution of F from complexes of projective modules that have zero differentials (see Section 1 for details). Projective level is a special case of a general notion of level in triangulated categories, which was introduced and studied by Avramov, Buchweitz, Iyengar, and Miller in [3]. It was partly motivated by earlier work of Christensen [8]; Bondal and Van den Bergh [7]; Dwyer, Greenlees, and Iyengar [9]; Rouquier [17]; and Krause and Kussin [16].

Upper bounds for projective level are relatively easy to obtain, since any explicit construction of a projective resolution provides such a bound; in particular, $\operatorname{level}^{\mathcal{P}}(F)$ does not exceed the number of non-zero modules in a projective resolution of F. This paper focuses on lower bounds.

In Section 2, we prove the following theorem which gives a lower bound on the projective level of a complex F in terms of the largest gap in the homology of F.

Theorem 2.1. *Let F be a complex of R-modules. Assume $\mathrm{H}_i(F) = 0$ for all $a < i < b$ with $a, b \in \mathbb{Z}$ and $\mathrm{H}_0(\Omega^R_{b-1}(F))$ is not projective. Then*

$$\operatorname{level}^{\mathcal{P}}(F) \geqslant b - a + 1.$$

Here, $\Omega^R_{b-1}(F)$ is the $(b-1)$th syzygy of F (see Section 1).

Restricting to commutative Noetherian local rings, in Section 3 we apply Theorem 2.1 to deduce a new version of The New Intersection Theorem. In the following result, $\operatorname{level}^R(F)$ is the smallest number of mapping cones needed to build F from finitely generated free R-modules.

Theorem 3.1. *Let R be a Noetherian local ring. Let*

$$F := 0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_0 \longrightarrow 0$$

be a complex of finitely generated free R-modules such that $\mathrm{H}_i(F)$ has finite length for every $i \geqslant 1$. For any ideal I that annihilates a minimal generator of $\mathrm{H}_0(F)$, there is an inequality

$$\operatorname{level}^R(F) \geqslant \dim(R) - \dim(R/I) + 1.$$

This result refines strong forms of the Improved New Intersection Theorem of Evans and Griffith [10], due to Bruns and Herzog [6] and Iyengar [15]. The proof uses the existence of balanced big Cohen–Macaulay algebras which was recently proved in [1]. Furthermore, in Theorem 3.2, we use Theorem 3.1 to prove that for every commutative Noetherian