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A generalization of intertwining operators for vertex operator algebras

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A B S T R A C T

For a vertex operator algebra V, we generalize the notion of an intertwining operator among an arbitrary triple of V-modules to an arbitrary triple of $\mathbb{N}$-graded weak V-modules and study their properties. We show a formula for the dimensions of the spaces of these intertwining operators in terms of modules over the Zhu algebras under some conditions on $\mathbb{N}$-graded weak modules.

1. Introduction

Let V be a vertex operator algebra. The purpose of this paper is to generalize the notion of an intertwining operator among an arbitrary triple of V-modules to an arbitrary triple of $\mathbb{N}$-graded weak V-modules. In the representation theory of groups or Lie algebras, the tensor product of two modules is the tensor product vector space whose module structure is defined by means of a natural coproduct operation, and an

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intertwining operator is defined to be a module homomorphism from the tensor product of two modules to a third module. In contrast, in the representation theory of vertex operator algebras, first the notion of an intertwining operator among an arbitrary triple of modules is defined in [7, Definition 5.4.1], and then the tensor product of two modules is defined by using intertwining operators. Note that the existence of the tensor product of two modules over a vertex operator algebra is not guaranteed in general. The dimension of the space of all intertwining operators among a triple of modules is called the *fusion rule*. It is a natural problem to determine fusion rules for a given vertex operator algebra. In [9, Theorem 1.5.3], Frenkel and Zhu give a formula for the fusion rule among an arbitrary triple of irreducible modules in terms of modules over the Zhu algebra as a generalization of a result in [20] for WZW models. Here we have to be careful that [9, Theorem 1.5.3] is correct for rational vertex operator algebras, however, is not correct for non-rational vertex operator algebras in general as pointed out at the end of [13, Section 2]. A modified result is given in [13, Theorem 2.11]. For many vertex operator algebras, fusion rules among triples of irreducible modules have been determined by using [9, Theorem 1.5.3] and [13, Theorem 2.11] (see for example [1], [2], [3], [16], [18], and [19]).

The definition of an intertwining operator in [7, Definition 5.4.1] makes sense even for weak modules, however, is no longer enough. Actually based on logarithmic conformal filed theories in physics, in [14] a generalization of the notion of an intertwining operator, called a *logarithmic intertwining operator*, is given among an arbitrary triple of logarithmic modules by allowing logarithmic terms. Here a logarithmic module is an $\mathbb{N}$-graded weak module such that each homogeneous space is a generalized $L(0)$-eigenspace. A generalization of the formula in [9, Theorem 1.5.3] and [13, Theorem 2.11] to logarithmic intertwining operators is given in [11, Theorem 6.6].

The aims of this paper are to introduce a generalization of the notion of an intertwining operator, which I call a $\mathbb{Z}$-*graded intertwining operator* (see Definition 3.2), among an arbitrary triple of general $\mathbb{N}$-graded weak modules and to study their properties. For logarithmic modules, a $\mathbb{Z}$-graded intertwining operator is essentially the same as a logarithmic intertwining operator as we will see later in Section 4. To explain the main idea, we recall a few facts about intertwining operators for (ordinary) modules over a vertex operator algebra $(V, Y, \mathbf{1}, \omega)$. For three V-modules $W_i = \oplus_{j=0}^{\infty}(W_i)_{\lambda_i+j}$ with lowest weight $\lambda_i \in \mathbb{C}$, $i = 1, 2, 3$ and an intertwining operator $I(\ ,x) : W_1 \otimes_{\mathbb{C}} W_2 \to W_3\{x\}$, we define an operator $I^o(u, x) = \sum_{i\in\mathbb{C}} u^o_i x^{-i-1} = x^{\lambda_1+\lambda_2-\lambda_3} I(\ ,x)$, which is already appeared in [9, (1.5.3)], [7, Remark 5.4.4], and [13, (2.12)]. Here $W_3\{x\} = \{\sum_{\alpha\in\mathbb{C}} w_\alpha x^\alpha \mid w_\alpha \in W_3\ (\alpha \in \mathbb{C})\}$. Then, $I^o(u, x)$ is a map from $W_1 \otimes_{\mathbb{C}} W_2$ to $W_3((x)) = \{\sum_{i\in\mathbb{Z}} w_i x^i \mid w_i \in W_3\ (i \in \mathbb{Z})$ and $w_i = 0,\ i \ll 0\}$ and $I(\ ,x)$ can be written as

$$I(u, x)v = \sum_{i\in\mathbb{Z}} x^{L(0)}(x^{-L(0)}u)^o_i x^{-L(0)}v \tag{1.1}$$

for $u \in W_1$ and $v \in W_2$. Here for the coefficient $L(0)$ of x^{-2} in each $Y_{W_i}(\omega, x)$, $i = 1, 2, 3$, we define