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Mixed tensors of the general linear supergroup

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ABSTRACT

We describe the image of the canonical tensor functor from Deligne's interpolating category $\underline{\mathrm{Rep}}(GL_{m-n})$ to $Rep(GL(m|n))$ attached to the standard representation. This implies explicit tensor product decompositions between any two projective modules and any two Kostant modules of $GL(m|n)$, covering the decomposition between any two irreducible $GL(m|1)$-representations. We also obtain character and dimension formulas. For $m > n$ we classify the mixed tensors with non-vanishing superdimension. For $m = n$ we characterize the maximally atypical mixed tensors and show some applications regarding tensor products.

1. Introduction

In this article we describe the indecomposable summands of the mixed tensor space $V^{\otimes r} \otimes (V^{\vee})^{\otimes s}$ $(r, s \in \mathbb{N})$ where $V = k^{m|n}$ is the standard representation of the General Linear Supergroup $GL(m|n)$ $(m \geq n)$ over an algebraically closed field k of characteristic zero. Such a summand is called a mixed tensor. These results imply decomposition laws for the tensor product between mixed tensors, character and dimension formulas and give

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us estimates about composition factors and Loewy lengths in tensor products between maximal atypical irreducible modules.

In the category of finite-dimensional algebraic representations $Rep(GL(m|n))$ the decomposition of the tensor product of two irreducible modules is known for a very small class of representations, the direct summands occurring in a tensor power of the standard representation $V \simeq k^{m|n}$. The tensor product decomposition is given by the Littlewood Richardson Rule. By [30], [4] the tensor space $V^{\otimes r}$ is completely reducible and the irreducible representations obtained in this way – the covariant representations – can be parametrized by $(m|n)$-hook partitions. It turns out that these representations form only a very small subset of the irreducible $GL(m|n)$-representations. In this article we look at the larger space of mixed tensors $V^{\otimes r} \otimes (V^\vee)^{\otimes s}$, $r, s \in \mathbb{N}$. Since $V^\vee$ is not unitary, this space is not fully reducible. The direct summands can be described via the Khovanov algebras of Brundan and Stroppel [9] and their tensor product decomposition can be understood using Deligne's interpolating categories. In [16] Deligne constructed for any $\delta \in k$ a karoubian rigid symmetric monoidal category $\underline{\mathrm{Rep}}(GL_\delta)$ which interpolates the classical representation categories $Rep(GL(n))$ in the sense that for $\delta = n \in \mathbb{N}$ we have an equivalence of tensor categories $\underline{\mathrm{Rep}}(GL_n)/\mathcal{N} \to Rep(GL(n))$ where $\mathcal{N}$ denotes the tensor ideal of negligible morphisms [1]. These interpolating categories possess a distinguished element of dimension δ which we call the standard representation st. Deligne's family of tensor categories are the universal tensor categories on a dualisable object of dimension δ in the sense of the universal Property 3.1.

In particular for $m - n \in \mathbb{N}_{\geq 0}$ we have two tensor functors starting from the Deligne category $\underline{\mathrm{Rep}}(GL_{m-n})$: One into $Rep(GL(m-n))$, the other one into $Rep(GL(m|n))$ (both determined by the choice of the standard representations $V = k^{m-n}$ respectively $V = k^{m|n}$). The tensor product decomposition in Deligne's category has been determined by Comes and Wilson [15]. If we are then able to understand the functor $F_{m|n} : \underline{\mathrm{Rep}}(GL_{m-n}) \to Rep(GL(m|n))$, $st \mapsto V$, we will be able to decompose tensor products in its image. Comes and Wilson also determine the kernel of the functor $F_{m|n}$ and show that its image is the space of mixed tensors T: The full subcategory of $Rep(GL(m|n))$ of objects which are direct summands in a tensor product $V^{\otimes r} \otimes (V^\vee)^{\otimes s}$ for some $r, s \in \mathbb{N}$. However Comes and Wilson do not describe the image $F_{m|n}(X)$ of an individual object X.

1.1. Main results

The space of mixed tensors has also been studied by Brundan and Stroppel [9]. In both approaches the indecomposable mixed tensors $R(\lambda)$ are described by certain pairs $\lambda = (\lambda^L, \lambda^R)$ of partitions, so-called $(m|n)$-cross bipartitions. The advantage of Brundan and Stroppels results is that they permit to analyze the Loewy structures of the mixed tensors and gives conditions on their highest weights. This allows to identify the image of an element under the tensor functor $\underline{\mathrm{Rep}}(GL_{m-n}) \to Rep(GL(m|n)$. In section 4 we define two invariants $d(\lambda)$ and $k(\lambda)$ of a bipartition.