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Solvable Lie algebras and graphs

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ABSTRACT

We define a solvable extension of the graph 2-step nilpotent Lie algebras of [5] by adding elements corresponding to the 3-cliques of the graph. We study some of their basic properties and we prove that two such Lie algebras are isomorphic if and only if their graphs are isomorphic. We also briefly discuss some metric properties, providing examples of homogeneous spaces with nonpositive curvature operator and solvsolitons.

1. Introduction

Attaching algebraic objects to combinatorial structures is a well-known tool to study properties both on the combinatorial, and on the algebraic side. Often it also has applications in geometry. One such example is [5], where Dani and Mainkar defined a 2-step nilpotent Lie algebra associated to any simple graph. Later, Mainkar [12] showed that

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such algebras are isomorphic if and only if the corresponding graphs are isomorphic. The construction has been used to study various geometric properties of the Lie algebras of graph type and, in particular, the existence of symplectic structures [14], possible nilsolitons and Einstein solvmanifolds defined by solvable extensions [10,9,7]. Often additional structures on graphs lead to similar constructions – in [15] a 3-step nilpotent metric Lie algebra has been associated to Schreier graph. Recently [13], the uniform nilpotent Lie algebras, which have extensions as Einstein solvmanifolds, have been encoded via a class of regular edge-colored graphs called uniformly colored graphs.

In this paper we initiate a study of solvable Lie algebras associated to another structure on a graph – the set of its 3-cliques (or 3-cycles). The definitions and constructions extend easily to k-cliques for any $k \geq 3$ and thus are related to one of the well-known NP-complete problems – the description of k-cliques in a graph. For a better clarity however, we focus on the case $k = 3$. This case also provides most of the examples relevant to the metric properties of the corresponding Lie group studied in Section 4.

The solvable Lie algebras constructed in the present paper are extensions of the nilpotent graph Lie algebras of [5] obtained by adding also elements corresponding to the 3-cliques with diagonal adjoint endomorphisms. In Section 2 we collect some of the basic properties of these algebras and, in particular, we characterize the center and the nilradical. Our main result (Section 3) is that two such algebras are isomorphic if and only if their defining graphs are isomorphic. The proof uses properties of maximal tori of the automorphism group, but its construction is slightly different from the one in the nilpotent case [12]. One of the main difficulties here is that the toral group generated by the automorphisms acting diagonally on the standard basis is no longer maximal and thus requires a more detailed analysis. In the last section, we describe some metric properties for solvable 3-clique graph algebras for which every vertex is adjacent to a 3-clique. They provide examples of homogeneous spaces of nonpositive curvature operator. The commutator of such algebras is the nilpotent algebra from [5] of the graph. Using this fact we also show that they define solvsolitons if and only if their derived algebra defines a nilsoliton, and that the natural basis we choose is stably Ricci diagonal. Finally we mention that the simply connected Lie groups of such algebras, endowed with left-invariant metrics, are isometric if and only if the corresponding graphs are isomorphic.

2. Cliques and solvable Lie algebras

In this section we define a Lie algebra associated to a graph, which extends the definitions in [5,12]. Recall that for a Lie algebra $\mathfrak{g}$ we have two descending series of subalgebras: $\mathfrak{g}^k = [\mathfrak{g}, \mathfrak{g}^{k-1}]$ and $\mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}]$, where $\mathfrak{g}^1 = \mathfrak{g}^{(1)} = \mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$. The Lie algebra $\mathfrak{g}$ is called *n-step nilpotent* if $\mathfrak{g}^n = 0$ but $\mathfrak{g}^{n-1} \neq 0$, and $\mathfrak{g}$ is called *n-step solvable* if $\mathfrak{g}^{(n)} = 0$, $\mathfrak{g}^{(n-1)} \neq 0$.

Let $\Bbbk$ be an arbitrary field. For a simple finite graph $\mathcal{G}$ with a set of vertices $V(\mathcal{G})$ numbered by $1, 2, \ldots, n$ and edges $E(\mathcal{G})$ denoted by (ij), we assign a vector e_i to each of vertex and we define the spaces